0. Read the remainder of the Nagel and Newman book (pages 92-113) and write a 300-500 word reflection. You might consider some of the following questions:

   (a) How does the proof outline in the book compare to what we are doing in class? Are there parts that are more or less clear, or things you now understand better?

   (b) What is the significance of the Incompleteness Theorems for mathematics as a discipline? How does (or should) this impact the way mathematicians think about what they are doing?

   (c) What questions do you still have regarding the Incompleteness Theorems? Does anything trouble you about the statement, proof or consequences? What are you still curious about?

   Remember that you can read other people's posts (and they can read yours).

1. Suppose that $S \subseteq \mathbb{N}^2$ is some set of *pairs of numbers*, and $T \subseteq \mathbb{N}$ is the set of sequences numbers of elements of S, i.e.

$$T = \{\langle a, b\rangle : \ (a, b) \in S\}.$$

   Prove that T is decidable if and only if S is decidable. You may freely use any facts stated in class, even if full proofs were not given (and you may also do this in all problems going forward). Be sure to use the definitions carefully, being attentive to the difference between a decidable set in $\mathbb{N}^2$ versus in $\mathbb{N}$.

2. Consider the following formula, with one free variable x:

$$\phi(x) \equiv (\exists x')((\neg x' = 0) \wedge (x' < x) \wedge (x = (\bar{2}E\bar{8}) \cdot (\overline{15}E(\bar{2}E((\bar{2} \cdot x') + \bar{1}) + 1)))).$$

   The red text was added later to correct a small bug. You can ignore it if you simply assume that all even numbers are symbol numbers of variables, including 0. See the email sent Tuesday afternoon. The issue is subtle and easy to miss, and no points will be deducted if you just ignore the issue.

   (a) Prove that $\phi(x)$ decides a set S of natural numbers.

   (b) Prove that every element of S is the Gödel number of some formula α. Describe, as simply as you can, which formulas these are.

3. Recall that a sentence ϕ in $\mathcal{L}_{NT}$ is *decidable* (from N) if either $N \vdash \phi$ or $N \vdash \neg\phi$. We make the following definition: a formula $\phi(x)$ with one free variable *decides a set* if for every $n \in \mathbb{N}$, the sentence $\phi(\overline{n})$ is decidable. Assume that $\phi(x)$ decides a set.

   (a) Explain informally why the sentence $(\exists x)\phi(x)$ may not be decidable. You don't need to give a proof, but give a plausible explanation.

   (b) Prove nonetheless that

$$\mathcal{N} \vDash (\exists x)\phi(x) \text{ if and only if } N \vdash (\exists x)\phi(x),$$

   and explain briefly why this doesn't contradict part (a).

4. Construct a (nonstandard) model of $\mathcal{L}_{NT}$ in which all the axioms N are true, but the sentence $(\exists x)(x + x = \overline{1})$ is also true (despite being false in the standard structure). Explain why the existence of this structure implies that the formula $\phi(x) \equiv (\exists x')(x' + x' = x)$ does not *decide* the set of even numbers, even though it *defines* it.

   Hint You are free to make this construction in any way you please. If you are stuck, though, here is a suggestion: let the universe be $\mathbb{R}_{\geq 0}$, the set of all nonnegative real numbers. This is larger than you need, but it will serve the purpose. Give all function symbols their standard interpretations. You will then need to choose an interpretation for $<$. The standard interpretation will not satisfy all axioms of N (which one fails?), so try to cook up an alternative. Make sure to check that is satisfies all the axioms of N.