Reading §1.3 through §1.5 of Leary-Kristiansen

This assignment concerns terms and formulas in a first-order language, and focuses on the crucial techniques of **induction** on the term or formula structure, and on the distinction between free and bound variables.

You should submit solutions to these problems on Gradescope. You may handwrite or type answers, but in any case they should be clearly and fully explained. Unless instructed otherwise, you should include proofs or explanations for any statements you make. Of course it's often hard to know how much detail to give, so please ask if you are unsure.

Problems:

1. Let $\mathcal{L}$ be a first-order language that has one constant symbol $\heartsuit$, one unary (i.e. 1-ary) function symbol $+$, and one ternary (i.e. 3-ary) relation symbol $>$.
   (a) Write down two different terms of $\mathcal{L}$ that have four symbols.
   (b) Write down two different formulas of $\mathcal{L}$ that have four symbols.
   (c) Write down a formula of $\mathcal{L}$ that includes the symbols $\neg$, $=$, and $+$.
   (d) Write down a formula of $\mathcal{L}$ that includes the symbols $\vee$, $\forall$, $\heartsuit$, and $>$.
2. Let $\mathcal{L}$ be the first-order language from #1. Use induction on the term structure to prove that there is no term in $\mathcal{L}$ that includes both the symbols $\heartsuit$ and x.
3. Let $\mathcal{L}$ be a first-order language with no relation symbols. Use induction on the formula structure to prove that every formula in $\mathcal{L}$ has more $=$ symbols than $\vee$ symbols.
4. Let $\mathcal{L}$ be the first-order language with constant symbol 0 and unary relation symbol $\star$.
   (a) For each of the four appearances of the variable x in the following formula, say whether that appearance is *free* or *bound*:
   $$((\forall x)(=x0) \vee ((\forall x)(\star 0) \vee \star x)).$$
   Justify your answer using the definition of free/bound appearances from class.
   (b) Write down a formula in the language $\mathcal{L}$ in which x and x' each appear both as free and bound variables. (As an added challenge, try to make your formula as short as possible subject to this condition.)
5. (Challenge, not for credit) Prove the *unique readability* property for terms described in Exercise #7 of Section 1.4. (Warning: the different way that I have introduced variables compared to the textbook means that the approach the book suggests doesn't exactly work.)