Note We will return to our usual Wednesday due date with this problem set. Since you have only five days to complete it, it is a bit shorter than usual.

1. Let $\mathcal{L}_{NT}$ be the language with constant symbol 0, unary function symbol S, binary function symbols $+, \cdot, E$ and binary relation symbol $>$. (This is the *language of number theory* described in Definition 1.5.1.)

   Let $\mathcal{N}$ be the *standard* structure for $\mathcal{L}_{NT}$ with universe $\mathbb{N}$ (note: the set of natural numbers includes 0 in this course), and in which the symbols have their 'usual' meanings (with S being the function that adds one to a natural number, and E standing for exponentiation):

   - $0^{\mathcal{N}} = 0$
   - $S^{\mathcal{N}}(x) = x + 1$, $+^{\mathcal{N}}(x, y) = x + y$, $\cdot^{\mathcal{N}}(x, y) = xy$, $E^{\mathcal{N}}(x, y) = x^y$
   - $>^{\mathcal{N}}(x, y) =$ true if $x > y$, false otherwise.

   (a) Let s be the variable assignment function for $\mathcal{N}$ that assigns to every variable the natural number 57. Calculate each of the following natural numbers:

      i. $\bar{s}(S{+}0x)$

      ii. $\bar{s}(ESx{\cdot}0x')$

   (b) Let s be the same variable assignment function as in part (a). Decide of each of the following formulas is true in $\mathcal{N}$ with s. (Explain your answers using Definition 1.7.4.)

      i. $=S0E0x$

      ii. $(\neg{>}xSx)$

      iii. $(=00 \vee \neg{=}00)$

      iv. $(\forall x)(= x'x'')$

   (c) Decide if the following formula is true in $\mathcal{N}$. (Recall that if we don't specify a vaf, then true means 'true with any vaf'.)

   $$((\forall x)(=x0) \vee (\forall x)(\neg{=}x0)).$$

   (d) Give an example of a structure for the language $\mathcal{L}_{NT}$ for which the answer to part (c) is different than for the structure $\mathcal{N}$. (For this part only, you don't need to prove your answer is correct. It is sufficient to describe the structure.)

2. Let $\mathcal{L}$ be a first-order language.

   (a) Write down a formula that is true in an $\mathcal{L}$-structure $\mathcal{A}$ if and only if the universe of $\mathcal{A}$ has exactly one element.

   (b) Write down a formula that is true in an $\mathcal{L}$-structure $\mathcal{A}$ if and only if the universe of $\mathcal{A}$ has exactly two elements.

   (In this problem, you should give brief informal explanations. You do not have to use Definition 1.7.4 to prove that your answers are correct.)

3. Prove that the formula $(\forall x)(=xx)$ is valid in any first-order language.