1. Recall that we defined first-order languages in a fairly minimalist way, which includes only two logical connectives $\neg, \vee$, and one quantifier $\forall$. This problem shows how the other common symbols $\wedge, \rightarrow, \exists$ can be defined in terms of these, and confirms that the semantics are as we would expect.

   (a) For a formula α and variable u in a first-order language $\mathcal{L}$, we will write $(\exists u)(\alpha)$ as short-hand for the formula
   $$(\neg(\forall u)((\neg\alpha))).$$
   (More informally, we can also omit some parentheses and write simply $\neg\forall u\neg\alpha$). Let s be a vaf for an $\mathcal{L}$-structure $\mathcal{A}$ with universe A. Prove that $(\exists u)(\alpha)$ is true in $\mathcal{A}$ with s if and only if there is $a \in A$ such that α is true in $\mathcal{A}$ with $s[u|a]$.

   (b) For two formulas α, β, we will write $\alpha \wedge \beta$ as short-hand for the formula $(\neg((\neg\alpha) \vee (\neg\beta)))$ (or, less formally, $\neg(\neg\alpha \vee \neg\beta)$). Prove that $\alpha \wedge \beta$ is true in $\mathcal{A}$ with s if and only if both α and β are true in $\mathcal{A}$ with s.

   (c) For two formulas α, β, we will write $\alpha \rightarrow \beta$ as short-hand for the formula $((\neg\alpha) \vee \beta)$. Prove that if $\alpha \rightarrow \beta$ and α and both true in $\mathcal{A}$ with s, then so is β.

2. Prove that the $\mathcal{L}_{NT}$-formula $(\forall x)(=+x0x)$ (i.e. more informally, the formula $(\forall x)(x+0=x)$) logically implies the formula $=+000$ (i.e. more informally, the formula $0+0=0$).

3. Let $\mathcal{L}$ be a first-order language, and let $\mathcal{A}$ be a structure for $\mathcal{L}$.

   (a) Let ϕ be a sentence in $\mathcal{L}$ (i.e. a formula with no free variables). Prove that ϕ is true in $\mathcal{A}$ if and only if $(\neg\phi)$ is not true in $\mathcal{A}$.

   (b) Give an example to show that the conclusion of part (a) does not necessarily hold if ϕ has a free variable.

4. Prove that all of the axioms (P1), (P2), (P3), (P4) from Friday's class are valid in any first-order language. That is, fix a first-order language $\mathcal{L}$, and let α, β, γ be formulas in $\mathcal{L}$. Prove that:

   (a) $\models \alpha \vee \alpha \rightarrow \alpha$

   (b) $\models \alpha \rightarrow \alpha \vee \beta$

   (c) $\models \alpha \vee \beta \rightarrow \beta \vee \alpha$

   (d) $\models (\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \alpha \vee \gamma)$

5. Write a deduction from the given premises to the given conclusion, in the propositonal calculus stated in Friday's class. That is, your deduction should use only the four axiom groups (P1), (P2), (P3), (P4), the given premises, and the one inference rule (MP).

   > **Note** For this problem, you should actually write a deduction in full, rather than simply proving that a deuction exists. Going forward we will often be content to prove that a deduction exists with various shortcuts, but for this problem you should write one out.

   (a) Premises: $\{\alpha \rightarrow \neg\beta,\ \beta\}$. Conclusion: $\neg\alpha$.

   (b) Premises: none. Conclusion: $\alpha \rightarrow \beta \vee \alpha$