0. Read Chapters I and II of *Gödel's Proof* (revised edition), by Nagel and Newman. You can find the full book electronically at a link on the Moodle Page (or get a paperback copy for about $10). After reading these chapters, write a short response (300-500 words) and post it on the forum linked on the course Moodle page. Here are some suggestions for questions to answer, but don't feel constrained by these, and you definitely don't want to answer them all. I much prefer a more in-depth response on one topic, than a short sentence about each.

   - What did you find surprising in the reading?
   - In what ways does the reading relate to what we have done in class?
   - What questions does the reading raise for you? What would you like you know about in relation to it?
   - What was new in the reading for you?
   - How has the reading changed the way you think about math?
   - What was difficult to understand about the reading?

   These posts are public so you will be able to read each others' responses.

1. (a) In each of the following cases, write down the formula α_t^u associated to the given formula α, variable u and term t (no explanations necessary):
      i. α is $(\forall x)(=xx')$, u is x, t is $+xx'$
      ii. α is $(\forall x)(=xx')$, u is x', t is $+xx'$
      iii. α is $((\forall x)(>xx') \vee (\forall x')(>xx'))$, u is x, t is $+x'x''$

   (b) In each of the cases in part (a), work out whether t is substitutable in α for u, or not. You should explain your answers clearly using the definition of 'substitutable'.

2. Write down a deduction for each of the following $\mathcal{L}_{NT}$-formulas, using the first-order logical axioms and rules of inference.

   (a) $Sx = Sx$
   (b) $((0 = 0) \vee \neg(0 = 0)) \to (Sx = Sx)$
   (c) $(\forall x)(Sx = Sx)$
   (d) $S0 = S0$

3. Let $\mathcal{L}$ be a first-order language with a unary relation symbol R. Write out an explicit deduction to show that

$$(\forall x)(Rx) \vdash (\forall x')(Rx').$$

4. Let Γ be a set of formulas in a first-order language $\mathcal{L}$. We make the following definition: a formula ϕ in $\mathcal{L}$ is called *decidable by* Γ if either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

   (a) Prove that if α is decidable by Γ, then so is $\neg\alpha$. (This doesn't sound like I'm saying anything at all, but there is something subtle to prove. Nonetheless, the proof is not long given what we've already established).

   (b) Prove that if α and β are both decidable by Γ, then so is $\alpha \vee \beta$.

   (c) Suppose that every atomic formula in $\mathcal{L}$ is decidable. Prove that any formula with no quantifiers is decidable.

> **Note** In the following problem, you should assume the Deduction Theorem for our full deductive system. It states: for any set of formulas Γ and any two formulas α and β, *if α is a sentence,* then $\Gamma, \alpha \vdash \beta$ if and only if $\Gamma \vdash \alpha \to \beta$.

5. Let Γ be a set of formulas in a first-order language $\mathcal{L}$. We make the following definitions: Γ is *inconsistent* if there is a formula ϕ such that both $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. Call Γ *explosive* if it deductively implies *every formula*, i.e. $\Gamma \vdash \phi$ for every formula ϕ in $\mathcal{L}$.

   (a) Prove that Γ is inconsistent if and only if Γ is explosive. Equivalently: if there is even a *single formula* ϕ such that $\Gamma \nvdash \phi$, then Γ is consistent.

   (b) Prove that if ϕ is a sentence, then $\Gamma \vdash \phi$ if and only if $\Gamma \cup \{\neg\phi\}$ is inconsistent.

   (c) Prove that if ϕ is a sentence, then ϕ is undecidable from Γ if and only if both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are consistent.