0. Read Chapter III of Gödel's proof (you do not need to submit a reading response for this one).

1. Problem 2.8.4 from the textbook (on formally writing down some of the axioms of Zermelo-Fraenkel set theory).

2. Read the statement of Lemma 2.8.4 in the textbook, and the paragraph above it for some notation. Part (2) of this lemma implies that

$$N \vdash \neg(= S0SS0).$$

Give an explicit abbreviated deduction of this fact.

3. We say that a set Σ of sentences in a first-order system is **complete** if for every sentence ϕ, we have either $\Sigma \vdash \phi$ or $\Sigma \vdash \neg\phi$.

   (a) Let $\mathcal{A}$ be a structure for a first-order language, and let $\text{Th}(\mathcal{A})$ be the **theory of** $\mathcal{A}$, that is, the set of sentences that are true in $\mathcal{A}$. Prove that $\text{Th}(\mathcal{A})$ is complete.

   (b) Let Σ be a consistent set of sentences with the property that for any set Σ' with $\Sigma \subseteq \Sigma'$, we have either $\Sigma = \Sigma'$, or Σ' is inconsistent. (We say that Σ is a **maximal consistent set of sentences**.) Prove that Σ is complete.

4. Prove that an infinite set of formulas Γ is consistent if and only if every *finite* subset of Γ is consistent.

5. Let $\mathcal{L}'_{NT}$ be an expansion of the language $\mathcal{L}_{NT}$ of number theory, in which one additional constant symbol c is added. Denote by $T = \text{Th}(\mathcal{N})$ be the theory (in the sense of the problem above) of the standard structure for $\mathcal{L}_{NT}$, which we will also regard as sentences of the expanded language $\mathcal{L}'_{NT}$. (Note that while T is complete in the original language, it is not complete in the expanded language, since it says nothing about c.)

   (a) Let ϕ_n denote the sentence $\overline{n} < c$, where $\overline{n} \equiv SS\cdots S0$ denotes the *canonical term* of the natural number n (as defined on p. 68). Prove that, for every n the set

$$T \cup \{\phi_0, \phi_1, \cdots, \phi_n\}$$

   is consistent.

   > **Hint** Describe a specific structure in which all these sentences are true. It's enough to take the structure $\mathcal{N}$ and add an interpretation of the constant c.

   (b) Prove that the set $T \cup \{\phi_i : i \in \mathbb{N}\}$, which includes *every* sentence ϕ_n, is also consistent.

   (c) Briefly explain why the result in part (b) is surprising or conterintuitive. (This is not a precise question, so it will not be graded strictly. Just think about it and try to see why something unsettling is being said here!)

6. Prove the following parts of the proof of the Soundness Theorem (whose proofs are omitted in the textbook). You should read the statements of the "two technical lemmas" in Section 2.6, and you may use these lemmas in your proofs. You do not need to carefully study the proofs of these technical lemmas.

   (a) Any logical axiom of the form (E3) is valid.

(b) Any logical axiom of the form (Q2) is valid.

(c) If the premise of a (QR2) rule is true in a structure $\mathcal{A}$, then the conclusion of that rule is true in $\mathcal{A}$.