0. The first part of this assignment is to continue reading the book by Nagel and Newman. The last chapter of this book (Chapter VII: Gödel's Proof) concerns the actual proof of the Incompleteness Theorem. Please read sections A and B of this chapter (pages 68-91). This sets the scene in a similar way to how we will approach it in class. Notice that there are lots of little differences (in the language, in the encoding process) but none of these really matter - the essential approach is the same. (You just have to read these sections. There is no reflection you need to write.)

1. We say that a formula ϕ in $\mathcal{L}_{NT}$, whose only free variable is x, *defines* a set $S \subseteq \mathbb{N}$ if for every $n \in \mathbb{N}$, we have $\mathcal{N} \models \phi^x_{\bar{n}}$ if and only if $n \in S$. Here $\mathcal{N}$ denotes the standard structure. Make sure you see the difference between "defines" and "decides!"

   For each of the following subsets of $\mathbb{N}$, find an $\mathcal{L}_{NT}$-formula ϕ that defines the given set. (You do not have to justify your answers.)

   (a) the set $\mathbb{N}$;
   (b) the set of odd numbers;
   (c) the set of square numbers;

2. Prove that any finite subset of $\mathbb{N}$ is decidable.

3. The First Incompleteness Theorem says that there is no set of $\mathcal{L}_{NT}$-formulas that is decidable, true-in-$\mathcal{N}$, and complete. For this problem, you don't need to use any formal definition of "decidable;" just use the informal definition of "able to be distinguished with a computer program." Give examples of sets of formulas that are:

   (a) decidable and true-in-$\mathcal{N}$
   (b) decidable and complete
   (c) true-in-$\mathcal{N}$ and complete

4. A variant of Gödel's Incompleteness Theorem, called *Rosser's Theorem*, says that there is no set of $\mathcal{L}_{NT}$-formulas that is decidable, consistent, contains N, and is complete. Again, in this problem, you don't need to use a formal definition of "decidable." Give examples of sets of formulas that are:

   (a) decidable, consistent, and contains N
   (b) decidable, consistent, and complete
   (c) decidable, contains N, and is complete
   (d) consistent, contains N, and is complete

5. Explain why the First Incompleteness Theorem implies that the set $\mathrm{Th}(\mathcal{N})$, that is, the set of all formulas true in the standard structure for number theory, is not decidable. (We will actually prove later that this set is not even definable.)