Reading Stewart §1.1-2.

1. Prove that every subfield $F \subseteq \mathbb{C}$ contains $\mathbb{Q}$.
2. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a field automorphism (i.e. an isomorphism from the field to itself), and assume that $\phi(a) = a$ for all $a \in \mathbb{R}$ (such ϕ is called a $\mathbb{R}$-automorphism of $\mathbb{C}$). Prove that $\phi(i)^2 + 1 = 0$, and therefore that $\phi(i) = \pm i$. Conclude that the only $\mathbb{R}$-automorphisms of $\mathbb{C}$ are the identity and complex conjugation.
3. Genearlizing the previous problem: let $F \subseteq \mathbb{C}$ be a subfield, and $\alpha \in \mathbb{C}$ a complex number such that $\alpha \notin F$ but $\alpha^2 \in F$. Prove that there are exactly two automorphisms $\phi : F(\alpha) \to F(\alpha)$ such that $\phi(a) = a$ for all $a \in F$.
4. Prove that $\{1, \sqrt{2}\}$ is a linearly independent set over $\mathbb{Q}$.
5. Denote by ζ_n the complex number $\zeta_n = e^{2\pi i/n}$. Prove that $\mathbb{Q}[\zeta_3] = \mathbb{Q}[\sqrt{-3}]$.
6. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.