1. (Textbook 17.4)
   ($K(t)$ as an S_3-extension)

2. Denote by $\overline{\mathbb{F}_p}$ an algebraic closure of $\mathbb{F}_p$. As discussed in class, for each $d \geq 1$, the set $\{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^d} = \alpha\}$ forms a subfield of exactly p^d elements, which we will denote $\mathbb{F}_{p^d}$.

   (a) For which d, e is $\mathbb{F}_{p^d}$ a subfield of $\mathbb{F}_{p^e}$?

   (b) Describe, as explicitly as possible, the Galois group of $\mathbb{F}_{p^e}/\mathbb{F}_{p^d}$ in the cases where $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^e}$.

3. Let $q = p^n$ denote a prime power, and let $\mathbb{F}_q$ be a field of q elements. For each $d \mid n$, denote by $\mathbb{F}_{p^d}$ the (unique) subfield of $\mathbb{F}_q$ of p^d elements.

   (a) Suppose that $\alpha \in \mathbb{F}_q$ has minimal polynomial $m \in \mathbb{F}_p[t]$ over $\mathbb{F}_p$, and let d be the degree of m. Prove that $d \mid n$ and $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^d}$.

   (b) In the notation of part (a), prove that m is an irreducible factor of $t^q - t$.

   (c) Prove that if m is *any monic irreducible polynomial* of degree d over $\mathbb{F}_p$, where $d \mid n$, then m splits in $\mathbb{F}_q$ and m is an irreducible factor of $t^q - t$.

   (d) Deduce that $t^q - t$ is equal to the product of all monic irreducible polynomials m over $\mathbb{F}_p$ such that $\partial m \mid n$.

4. Let p be a prime number. For all $d \geq 1$, denote by a_d the number of irreducible polynomials over $\mathbb{F}_p$ of degree d.

   (a) Prove that for all $n \geq 1$, the following formula holds.

$$p^n = \sum_{d \mid n} d \; a_d$$

   > **Hint** Use the previous problem. You can assume its results are true even if you haven't solved it yet.

   (b) Find a formula for a_{10} in terms of p.