1. (Textbook 9.1 and 9.2)
   Determine splitting fields over $\mathbb{Q}$ for the polynomials $t^3 - 1, t^4 + 5t^2 + 6, t^6 - 8$, in the form $\mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ for explicit α_j, and determine the degree over $\mathbb{Q}$ in each case.

2. (Textbook 9.5 (b-e))
   Which of the following extensions are Galois? (The statement in the book says "normal" here, but this is equivalent to "normal" for subfields of $\mathbb{C}$.)

   (b) $\mathbb{Q}(\sqrt{-5})/\mathbb{Q}$

   (c) $\mathbb{Q}(\alpha)/\mathbb{Q}$ where α is the real seventh root of 5.

   (d) $\mathbb{Q}(\sqrt{5}, \alpha)/\mathbb{Q}(\alpha)$, where α is as in (c).

   (e) $\mathbb{R}(\sqrt{-7})/\mathbb{R}$

3. Determine the Galois group of the splitting field of $t^3 - 3t + 1$ over $\mathbb{Q}$.

4. Suppose that $f \in K[t]$ is a quartic polynomial over a subfield $K \subseteq \mathbb{C}$, with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $\mathbb{C}$. Suppose that the symmetric group of f over K consists of the following four permutations (these form the Klein 4-group).

   $$\{\mathrm{id},\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

   Here, we identify the roots with the numbers $1, 2, 3, 4$. Define $u_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4$, $u_3 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4$, and $u_4 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4$. Prove that u_2, u_3, u_4 are eigenvectors for each element of the Galois group, with eigenvalues ± 1. Conclude that $u_2^2, u_3^2, u_4^2 \in K$, and hence that the splitting field of f is a radical extension of K.

   Note This seemingly ad hoc construction can be generalized to any abelian Galois group as follows. For a finite abelian group G, define the *dual group* $\hat{G}$ to be the group of "characters," i.e. group homomorphisms $\chi : G \to \mathbb{C}^*$. Fix a root α of the polynomial in question, and define elements $u_\chi = \sum_{g \in G} \chi(g)\, g(\alpha)$. Then one can show that $g(u_\chi) = \chi(g)^{-1}\, u_\chi$, and the elements u_χ generate all the roots in the orbit of α. The Fourier transform we used while discussing the cubic formula is another example of this construction.