MATH 42
FINAL EXAM
11 MAY 2015

Name : ____

- The time limit is 3 hours.
- No calculators or notes are permitted.
- The last page is a multiplication table for arithmetic modulo 29, which will be useful for several problems. You may detach it from the packet for ease of use if you wish.

1	/20	**2**	/5	**3**	/5
4	/5	**5**	/5	**6**	/5
7	/5	**8**	/5	**9**	/5
10	/6	**11**	/7	**12**	/7
				$\sum$	/80

(1) **Short answer questions.** Each answer is worth 2 points. You do not need to show any work. **Several questions have multiple possible answers; you only need to give one.**

(a) Compute the greatest common divisor of 77 and 91.

Answer: __________

(b) Find a perfect number (that is, a positive number which is equal to twice the sum of all of its divisors, including 1 and itself).

Answer: __________

(c) Find an integer x such that $3x \equiv 4 \pmod{7}$.

Answer: __________

(d) Find the smallest *positive* number of the form $15x + 39y$, where x and y are integers (positive or negative).

Answer: __________

(e) Find a positive integer n such that $10^n \equiv 1 \pmod{113}$. (The number 113 is prime)

Answer: __________

(f) Evaluate $\phi(130)$.

Answer: __________

(g) Find an integer x, between 0 and 28 inclusive, such that $x^2 \equiv -1 \pmod{29}$. (You may wish to use the multiplication table on the last page.)

Answer: __________

(h) Evaluate the Legendre symbol $\left(\frac{-2}{37}\right)$.

Answer: __________

(i) Find a primitive root of 7.

Answer: __________

(j) Find an integer n, greater than 100, which is *not* a sum of two squares (the number 0 is considered a square).

Answer: __________

(20 points)

(2) Solve the following congruence.

$$123x \equiv 3 \pmod{301}$$

Your answer should be in the form $x \equiv a \pmod{m}$, where a is between 0 and $m - 1$ inclusive.

(5 points)

(3) Solve the following pair of congruences.

$$\begin{aligned} x &\equiv 3 \pmod{15} \\ x &\equiv 13 \pmod{16} \end{aligned}$$

Your answer should be a *single* congruence of the form
$x \equiv a \pmod{m}$, where a is between 0 and $m-1$ inclusive.

(5 points)

(4) For each of the following four numbers (with factorization into primes given), either write the number as a sum of two squares or state that it is impossible to do so.

(a) $962 = 2 \cdot 13 \cdot 37$

(b) $1189 = 29 \cdot 41$

(c) $1725 = 3 \cdot 5^2 \cdot 23$

(d) $6137 = 17 \cdot 19^2$

(5 points)

(5) Prove that $\sqrt{7}$ is irrational.

(5 points)

(6) (a) List all of the prime numbers between 70 and 100.

(b) For which of these prime numbers p does $x^2 \equiv 5 \pmod{p}$ have an integer solution x?

(c) For which of these prime numbers p does $x^2 \equiv 3 \pmod{p}$ have an integer solution x?

(5 points)

(7) You are trying to read a certain 5-digit number on a piece of paper, but two of the digits are illegible. What you can read is the following (the units and hundreds digits are illegible).

57_3_

Fortunately, you know two facts about this number:

- It is divisible by both 4 and 9.
- All five digits are different.

Determine the number.

(5 points)

(8) Suppose that a, e, f, and m are positive integers such that the following two congruences hold.

$$\begin{aligned} a^e &\equiv 1 \pmod{m} \\ a^f &\equiv 1 \pmod{m} \end{aligned}$$

Prove that

$$a^{\gcd(e,f)} \equiv 1 \pmod{m}.$$

(5 points)

(9) Solve the congruence

$$x^{23} \equiv 5 \pmod{29}.$$

Your answer should be in the form $x \equiv a \pmod{m}$, where a is between 0 and $m-1$ inclusive.

(You may want to use the multiplication table on the last page.)

Hint. The answer will be congruent to 5^f for a well-chosen value of f.

(5 points)

(10) Consider the rather large number $N = 2^{53^{69}}$ (Note that this is 2 raised to the power 53^{69}, not 2^{53} raised to the power 69.)

(a) Find the remainder when N is divided by 4.

(b) Find the remainder when N is divided by 25.

(c) From parts (a) and (b), deduce the last two digits (units digit and tens digit) of N.

(6 points)

(11) Alice has a message m, encoded as a number between 0 and 28 inclusive, which she wishes to communicate to you using ElGamal encryption[1]. As part of your secret key, you know the following fact.

$$19^{10} \equiv 6 \pmod{29}$$

Alice has generated a number a, which she keeps secret, but she guarantees that the following two congruences are true.

$$\begin{aligned} 19^a &\equiv 7 \pmod{29} \\ m \cdot 6^a &\equiv 10 \pmod{29} \end{aligned}$$

From this information, recover the number m.

(You may wish to use the multiplication table on the last page.)

Hint. It is possible to compute m *without* computing the number a.

(7 points)

[1]You do not need any specific knowledge of ElGamal keys and encryption to solve the problem; the three congruences given are enough to solve for m.

(12) Prove that the equation

$$a^2 + b^2 = 3$$

has no *rational* solutions (i.e. there are no two rational numbers a, b satisfying the equation).

(7 points)

(additional space for work)

Multiplication table modulo 29

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	1	3	5	7	9	11	13	15	17	19	21	23	25	27
3	0	3	6	9	12	15	18	21	24	27	1	4	7	10	13	16	19	22	25	28	2	5	8	11	14	17	20	23	26
4	0	4	8	12	16	20	24	28	3	7	11	15	19	23	27	2	6	10	14	18	22	26	1	5	9	13	17	21	25
5	0	5	10	15	20	25	1	6	11	16	21	26	2	7	12	17	22	27	3	8	13	18	23	28	4	9	14	19	24
6	0	6	12	18	24	1	7	13	19	25	2	8	14	20	26	3	9	15	21	27	4	10	16	22	28	5	11	17	23
7	0	7	14	21	28	6	13	20	27	5	12	19	26	4	11	18	25	3	10	17	24	2	9	16	23	1	8	15	22
8	0	8	16	24	3	11	19	27	6	14	22	1	9	17	25	4	12	20	28	7	15	23	2	10	18	26	5	13	21
9	0	9	18	27	7	16	25	5	14	23	3	12	21	1	10	19	28	8	17	26	6	15	24	4	13	22	2	11	20
10	0	10	20	1	11	21	2	12	22	3	13	23	4	14	24	5	15	25	6	16	26	7	17	27	8	18	28	9	19
11	0	11	22	4	15	26	8	19	1	12	23	5	16	27	9	20	2	13	24	6	17	28	10	21	3	14	25	7	18
12	0	12	24	7	19	2	14	26	9	21	4	16	28	11	23	6	18	1	13	25	8	20	3	15	27	10	22	5	17
13	0	13	26	10	23	7	20	4	17	1	14	27	11	24	8	21	5	18	2	15	28	12	25	9	22	6	19	3	16
14	0	14	28	13	27	12	26	11	25	10	24	9	23	8	22	7	21	6	20	5	19	4	18	3	17	2	16	1	15
15	0	15	1	16	2	17	3	18	4	19	5	20	6	21	7	22	8	23	9	24	10	25	11	26	12	27	13	28	14
16	0	16	3	19	6	22	9	25	12	28	15	2	18	5	21	8	24	11	27	14	1	17	4	20	7	23	10	26	13
17	0	17	5	22	10	27	15	3	20	8	25	13	1	18	6	23	11	28	16	4	21	9	26	14	2	19	7	24	12
18	0	18	7	25	14	3	21	10	28	17	6	24	13	2	20	9	27	16	5	23	12	1	19	8	26	15	4	22	11
19	0	19	9	28	18	8	27	17	7	26	16	6	25	15	5	24	14	4	23	13	3	22	12	2	21	11	1	20	10
20	0	20	11	2	22	13	4	24	15	6	26	17	8	28	19	10	1	21	12	3	23	14	5	25	16	7	27	18	9
21	0	21	13	5	26	18	10	2	23	15	7	28	20	12	4	25	17	9	1	22	14	6	27	19	11	3	24	16	8
22	0	22	15	8	1	23	16	9	2	24	17	10	3	25	18	11	4	26	19	12	5	27	20	13	6	28	21	14	7
23	0	23	17	11	5	28	22	16	10	4	27	21	15	9	3	26	20	14	8	2	25	19	13	7	1	24	18	12	6
24	0	24	19	14	9	4	28	23	18	13	8	3	27	22	17	12	7	2	26	21	16	11	6	1	25	20	15	10	5
25	0	25	21	17	13	9	5	1	26	22	18	14	10	6	2	27	23	19	15	11	7	3	28	24	20	16	12	8	4
26	0	26	23	20	17	14	11	8	5	2	28	25	22	19	16	13	10	7	4	1	27	24	21	18	15	12	9	6	3
27	0	27	25	23	21	19	17	15	13	11	9	7	5	3	1	28	26	24	22	20	18	16	14	12	10	8	6	4	2
28	0	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1