MATH 42
MIDTERM 1
20 FEBRUARY 2015

Name : Solutions

- The time limit is 50 minutes.
- No calculators or notes are permitted.
- Each problem is worth 5 points.

1	/5	2	/5
3	/5	4	/5
5	/5	6	/5
		$\sum$	/30

(1) Find all prime numbers p between 1 and 100 such that

$$p \equiv -1 \pmod{15}.$$

The numbers congruent to -1 mod 15 between 1 and 100 are:

14, 29, 44. 59, 74, 89

Of these, 3 are even. But the rest turn out to be prime.

$\boxed{29, 59, 89}$

(2) Recall that a *primitive Pythagorean triple* consists of three positive integers (a, b, c) such that
- $a^2 + b^2 = c^2$, and
- there are no common factors of a, b and c.

Find a primitive Pythagorean triple such that $a = 15$.

We wish to solve

$$15^2 + b^2 = c^2$$

i.e. $$15^2 = (c+b)(c-b).$$

Since c, b should have no common factors, neither should $c+b$, $c-b$.
Therefore there are two ways to find solutions:

$$\begin{cases} c+b = 5^2 \\ c-b = 3^2 \end{cases} \Rightarrow \begin{cases} c = \frac{25+9}{2} = 17 \\ b = \frac{25-9}{2} = 8 \end{cases}$$

$\underline{(a,b,c) = (15, 8, 17)}$

$$\begin{cases} c+b = 15^2 \\ c-b = 1 \end{cases} \Rightarrow \begin{cases} c = \frac{225+1}{2} = 113 \\ b = \frac{225-1}{2} = 112 \end{cases}$$

$\underline{(a,b,c) = (15, 112, 113)}$

Both are PPT's, and in fact these are the only possible choices.

<u>Remark.</u> Both can be found easily by the recipe $(st, \frac{s^2-t^2}{2}, \frac{s^2+t^2}{2})$.
The logic above follows the logic of how this recipe was found.

(3) Compute the greatest common divisor of 1106 and 203.

$$1106 - 5 \cdot 203 = 1106 - 1015$$
$$= 91$$
$$203 - 2 \cdot 91 = 203 - 182$$
$$= 21$$
$$91 - 4 \cdot 21 = 91 - 84$$
$$= 7$$
$$21 - 3 \cdot 7 = 0.$$

Hence $\gcd(1106, 203)$
$$= \gcd(203, 91)$$
$$= \gcd(91, 21)$$
$$= \gcd(21, 7)$$
$$= \gcd(7, 0) = \underline{\underline{7}}.$$

(4) Solve the following congruence.

$$28x \equiv 3 \pmod{149}$$

Applying the (extended) Euclidean algorithm to 28 and 149:

$$
\begin{aligned}
&(149) \quad (28) \\
&[9] = (149) - 5\cdot(28) \\
&[1] = (28) - 3\cdot[9] = (28) - 3\cdot[(149) - 5\cdot(28)] \\
&\quad = 16\cdot(28) - 3(149).
\end{aligned}
$$

Thus $16\cdot 28 \equiv 1 \bmod 149$. Therefore

$$28x \equiv 3 \pmod{149}$$

$$\Updownarrow$$

$$16\cdot 28x \equiv 16\cdot 3 \bmod 149$$

$$\Updownarrow$$

$$\boxed{x \equiv 48 \bmod 149}$$

(5) Suppose that a, b, c are positive integers such that $\gcd(a, b) = 1$. Prove that if a divides bc, then a divides c.

Solution 1

Since $\gcd(a,b)=1$, there exist $x,y \in \mathbb{Z}$ s.t. $ax+by=1$.
Then

$$cax + cby = c$$

$$a\cdot\left(cx + \frac{bc}{a}\cdot y\right) = c$$

Since $a \mid bc$, $\frac{bc}{a} \in \mathbb{Z}$. So $cx + \frac{bc}{a}\cdot y \in \mathbb{Z}$, so c is a multiple of a.

Solution 2

Make use of unique factorization into primes. Then a, b, c can be written

$$\left.\begin{aligned} a &= p_1 p_2 \cdots p_\ell \\ b &= q_1 q_2 \cdots q_m \\ c &= r_1 r_2 \cdots r_n \end{aligned}\right\} \text{each } p_i, q_i, r_i \text{ prime. (not necessarily distinct).}$$

Then ~~the~~ $a \mid bc$ means that $\frac{bc}{a} \in \mathbb{Z}$; it too can be factored into primes. By uniqueness this factorization includes $q_1, \cdots, q_m, r_1, \cdots, r_n$ except that each $p_1, \cdots, p_\ell$ has been removed from the list. Since $\gcd(a,b)=1$, no prime p_i equals a prime q_j; so in fact each p_i occurs (the right number of times) in the factorization of c alone. Hence $a \mid c$.

(6) Suppose that you enter a store carrying a large supply of 6 dollar coins. The shopkeeper is able to make change using 28 dollar coins and 63 dollar coins. Find a way that you can purchase a 1 dollar item.

For partial credit, you may first assume that both you and the shopkeeper have a large supply of all three types of coins (6,28, and 63) and solve the problem in this context.

This amounts to solving

$$6x = 28y + 63z + 1.$$

For full credit we require $x, y, z \geq 0$. For partial credit, they can be negative integers as well. I saw a number of nice solutions; here are a couple.

Solution 1 Try to choose y, z first. As long as $6 \mid (28y + 63z + 1)$, there will be a working choice of x. So we require

$$28y + 63z \equiv -1 \bmod 6$$

$$\iff \quad -2y + 3z \equiv -1 \bmod 6 \quad \text{(reducing 28 \& 63)}$$

One sol'n is $y = 2, z = 1$. Then we should take

$$x = \frac{28 \cdot 2 + 63 \cdot 1 + 1}{6} = \frac{120}{6} = 20.$$

So you can pay 20 \$6 coins and receive 2 \$28 coins and 1 \$63 coin in change.

Solution 2 Notice that you can:

a) • Pay \$2: by giving 5 \$6 coins and getting 1 \$28 back

b) • Get \$3 back: by giving 10 \$6 coins and getting 1 \$63 coin back.

So you can do (a) twice and then do (b) once.

This amounts to giving 20 \$6 coins and getting 2 \$28 coins and a \$63 coin back.

In general One can prove that all solutions (positive & negative) have the form:

$$\left.\begin{aligned} x &= 20 + 14k + 21h \\ y &= 2 + 3k \\ z &= 1 + 2h \end{aligned}\right\} \text{ for } k, h \in \mathbb{Z}.$$

I often saw these choices:

(20, 2, 1)

(76, 5, 5)

(160, 32, 1)

(additional space for work)