$\boxed{\text{P. Set 10 Solutions}}$

① a) $x^2 \equiv -1 \bmod 5987$ has a sol'n $\Leftrightarrow$ $\left(\frac{-1}{5987}\right) = 1$.

Now, $\left(\frac{-1}{5987}\right) = (-1)^{\frac{1}{2}\cdot 5986} = -1$. so there is $\boxed{\text{no solution}}$.

b) $x^2 \equiv 6780$ has a solution $\Leftrightarrow$ $\left(\frac{6780}{6781}\right) = 1$.

Now, $\left(\frac{6780}{6781}\right) = \left(\frac{-1}{6781}\right) = 1$ since $6781 \equiv 1 \bmod 4$. So $\boxed{\text{a solution exists}}$.

c) $x^2 + 14x - 35 \equiv 0 \bmod 337$

$\Leftrightarrow$ $(x+7)^2 - 35 - 49 \equiv 0 \bmod 337$

$\Leftrightarrow$ $(x+7)^2 \equiv 84 \bmod 337$

So a solution exists if and only if $u^2 \equiv 84 \bmod 337$ can be solved, ie. we must calculate $\left(\frac{84}{337}\right)$.

$$\left(\frac{84}{337}\right) = \left(\frac{4}{337}\right)\cdot\left(\frac{3}{337}\right)\cdot\left(\frac{7}{337}\right)$$

$$= \left(\frac{2}{337}\right)^2 \cdot \left(\frac{337}{3}\right)\left(\frac{337}{7}\right) \quad \text{(quad. reciprocity)}$$

$$= 1\cdot\left(\frac{1}{3}\right)\cdot\left(\frac{1}{7}\right) = 1.$$ $\boxed{\text{So a solution exists}}$.

d) $x^2 - 64x + 943 \equiv 0 \bmod 3011$

$\Leftrightarrow$ $(x-32)^2 - 1024 + 943 \equiv 0 \bmod 3011$

$\Leftrightarrow$ $(x-32)^2 \equiv 81 \bmod 3011$ (note 81 is already a square)

$\Leftrightarrow$ $x - 32 \equiv \pm 9 \bmod 3011$

$\Leftrightarrow$ $x \equiv 23$ or 41

In particular, $\boxed{\text{a solution exists}}$.

②

a) $\left(\frac{85}{101}\right) = \left(\frac{101}{85}\right)$ (Q. recip: $101 \equiv 1 \bmod 4$)

$= \left(\frac{16}{85}\right)$ $(101 \equiv 16 \bmod 85)$

$= \left(\frac{4}{85}\right)^2 = \boxed{1}$

b) $\left(\frac{29}{541}\right) = \left(\frac{541}{29}\right)$ $(29 \equiv 1 \bmod 4)$

$= \left(\frac{19}{29}\right)$ $(541 \equiv 19 \bmod 29)$

$= \left(\frac{29}{19}\right)$ $(29 \equiv 1 \bmod 4)$

$= \left(\frac{10}{19}\right)$

$= \left(\frac{2}{19}\right) \cdot \left(\frac{5}{19}\right)$

$= (-1) \cdot \left(\frac{5}{19}\right)$ $(19 \equiv 3 \bmod 8)$

$= -\left(\frac{19}{5}\right)$ $(5 \equiv 1 \bmod 4)$

$= -\left(\frac{4}{5}\right) = -\left(\frac{2}{5}\right)^2 = \boxed{-1}.$

c) $\left(\frac{101}{1987}\right) = \left(\frac{1987}{101}\right)$

$= \left(\frac{68}{101}\right) = \underbrace{\left(\frac{2}{101}\right) \cdot \left(\frac{2}{101}\right)}_{\cancel{\text{squa}}\ 1} \cdot \left(\frac{17}{101}\right)$

$= \left(\frac{101}{17}\right)$ $(101 \equiv 1 \bmod 4)$

$= \left(\frac{-1}{17}\right) = \boxed{1}$ $(17 \equiv 1 \bmod 4)$.

d) $\left(\frac{31706}{43789}\right) = \left(\frac{2}{43789}\right) \cdot \left(\frac{15853}{43789}\right)$

$= -\left(\frac{15853}{43789}\right)$ $(43789 \equiv 5 \bmod 8)$

$= -\left(\frac{43789}{15853}\right)$

$= -\left(\frac{12083}{15853}\right)$ $(43789 \equiv 12083 \bmod 15853)$

$$= -\left(\frac{15853}{12083}\right) \qquad (15853 \equiv 1 \bmod 4)$$

$$= -\left(\frac{3770}{12083}\right)$$

$$= -\left(\frac{2}{12083}\right)\left(\frac{1885}{12083}\right)$$

$$= -(-1)\cdot\left(\frac{1885}{12083}\right) \qquad (12083 \equiv 3 \bmod 8)$$

$$= +\left(\frac{12083}{1885}\right) = \left(\frac{773}{1885}\right) \qquad (12083 \equiv 773 \bmod 1885)$$

$$= \left(\frac{1885}{773}\right) \qquad (1885 \equiv 1 \bmod 4)$$

$$= \left(\frac{339}{773}\right) \qquad (1885 \equiv 339 \bmod 773)$$

$$= \left(\frac{773}{339}\right) \qquad (773 \equiv 1 \bmod 4$$

$$= \left(\frac{95}{339}\right) \qquad (773 \equiv 95 \bmod 339)$$

$$= -\left(\frac{339}{95}\right) \qquad (95 \equiv 339 \equiv 3 \bmod 4, \text{ so reciprocity adds a } -)$$

$$= -\left(\frac{54}{95}\right)$$

$$= -\left(\frac{2}{95}\right)\cdot\left(\frac{27}{95}\right)$$

$$= -\underbrace{(1)}_{\substack{\text{since}\\ 95 \equiv 7 \bmod 8}}\cdot\underbrace{\left(-\left(\frac{95}{27}\right)\right)}_{\text{since } 27, 95 \text{ are both } 3 \bmod 4}$$

$$= \left(\frac{95}{27}\right) = \left(\frac{14}{27}\right) = \left(\frac{2}{27}\right)\cdot\left(\frac{7}{27}\right)$$

$$= \underbrace{(-1)}_{27 \equiv 3 \bmod 8}\cdot\underbrace{\left(-\left(\frac{27}{7}\right)\right)}_{7, 27 \text{ both } 3 \bmod 4}$$

$$= \left(\frac{27}{7}\right) = \left(\frac{6}{7}\right) = \left(\frac{-1}{7}\right)$$

$$= \boxed{-1} \quad \text{since } 7 \equiv 3 \bmod 4.$$

(3) Suppose $n+5=x^2$. Then for any prime factor p of n.

$$5 \equiv x^2 \mod p.$$

~~As~~ As long as $p \neq 2$ (p is an odd prime) and $p \neq 5$ (so that $5 \not\equiv 0 \mod p$), it follows from quadratic reciprocity that:

$$\left(\frac{5}{p}\right) = 1$$

$$\Rightarrow \quad \left(\frac{p}{5}\right) = 1 \qquad (\text{since } 5 \equiv 1 \bmod 4)$$

$$\Rightarrow \quad p \equiv y^2 \bmod 5 \quad \text{for some } y \text{ not divis. by } 5.$$

Hence either $p \equiv 1$ or $4 \mod 5$ since these are the quadratic residues of 5.

(4) By quadratic reciprocity,

$$\left(\frac{3}{p}\right) = (-1)^{\frac{(p-1)(3-1)}{4}} \cdot \left(\frac{p}{3}\right)$$

$$= (-1)^{\frac{p-1}{2}} \cdot \left(\frac{p}{3}\right).$$

Now, this shows that $\left(\frac{3}{p}\right)=1$ if and only if

either $(-1)^{\frac{p-1}{2}}$ & $\left(\frac{p}{3}\right)$ are both $+1$
or $(-1)^{\frac{p-1}{2}}$ & $\left(\frac{p}{3}\right)$ are both -1.

Now, given that p is odd and not 3,

$$(-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4 \\ -1 & \text{if } p \equiv 3 \bmod 4 \end{cases} \qquad \left(\frac{p}{3}\right) = \begin{cases} 1 & p \equiv 1 \bmod 3 \\ -1 & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

Hence $\left(\frac{3}{p}\right)=1$ if and only if

$$\left(\begin{array}{ll} \text{either} & p \equiv 1 \bmod 4 \ \& \ p \equiv 1 \bmod 3 \\ \text{or} & p \equiv 3 \bmod 4 \ \& \ p \equiv 2 \bmod 3 \end{array} \right) \underset{\text{chinese remainder theorem}}{\Longleftrightarrow} \boxed{\begin{array}{ll} \text{either} & p \equiv 1 \bmod 12 \\ \text{or} & p \equiv 11 \bmod 12 \end{array}}$$