$\boxed{\text{Worksheet for 11/5/13}}$

Evaluate the following limits:

1. $$\lim_{x\to\infty} \frac{\ln x}{\sqrt[10]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{10}x^{-9/10}} = \lim 10\cdot\frac{1}{x^{1/10}} = \boxed{0.}$$
$$\frac{\infty}{\infty}$$

2. $$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim \frac{e^x + e^{-x}}{\cos x} = \boxed{2}$$
$$\frac{0}{0}$$

3. $$\lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \boxed{2}$$

4. $$\lim_{x\to 0} \frac{\sin x}{\tan^{-1} x} = \lim \frac{\cos x}{1/(1+x^2)} = \boxed{1}$$
$$\frac{0}{0}$$

5. $$\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}\cdot x)}{\sqrt{x} - 1} = \lim_{x\to 1} \frac{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)}{1/2\sqrt{x}} = \boxed{-\pi.}$$
$$\frac{0}{0}$$

⑥ $\lim\limits_{x\to\infty} \dfrac{e^x}{x^4} = \lim \dfrac{e^x}{4x^3} = \lim \dfrac{e^x}{12x^2} = \lim \dfrac{e^x}{24x} = \lim \dfrac{e^x}{24} = \boxed{\infty.}$

$\quad \frac{\infty}{\infty} \quad \frac{\infty}{\infty} \quad \frac{\infty}{\infty} \quad \frac{\infty}{\infty}$

⑦ $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{\sqrt{x}} = \lim \dfrac{\frac{1}{x\ln x}}{\frac{1}{2\sqrt{x}}} = \lim \dfrac{2\sqrt{x}}{x} = \lim \dfrac{2}{\sqrt{x}} = \boxed{0}$

$\quad \frac{\infty}{\infty}$

⑧ $\lim\limits_{x\to 0} \dfrac{x^3}{x-\tan^{-1}x} = \lim \dfrac{3x^2}{1-\frac{1}{1+x^2}} = \lim \dfrac{3x^2}{\frac{x^2}{1+x^2}} = \lim \dfrac{3\cancel{x^2}(1+x^2)}{\cancel{x^2}} = \boxed{3.}$

$\quad \frac{0}{0}$

⑨ $\lim\limits_{x\to \pi} \dfrac{\sin x+x-\pi}{(x-\pi)^3} = \lim \dfrac{\cos x+1}{3(x-\pi)^2} = \lim \dfrac{-\sin x}{6(x-\pi)} = \lim \dfrac{-\cos x}{6} = \boxed{\frac{1}{6}.}$

$\quad \frac{0}{0} \quad \frac{0}{0} \quad \frac{0}{0}$

⑩ $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2} = \lim \dfrac{e^x-1}{2x} = \lim \dfrac{e^x}{2} = \boxed{\frac{1}{2}.}$

$\quad \frac{0}{0} \quad \frac{0}{0}$

Part II

[and current PSet #1]

1. (cf: PSet 8, #5). Show with examples why the following forms are indeterminate.

a) 0/0 $\quad \frac{2x}{x} \to 2, \quad \frac{x}{2x} \to \frac{1}{2} \quad$ as $x \to 0$

$\frac{\sin x}{x} \to 1 \quad \frac{\sin x}{\sqrt{x}} \to 0 \quad$ as $x \to 0$

b) $\infty/\infty \quad \frac{2x}{x} \to 2 \quad \frac{x}{2x} \to \frac{1}{2} \quad$ as $x \to \infty$

$\frac{e^x}{x} \to \infty \quad \frac{x}{e^x} \to 0 \quad$ as $x \to \infty$

c) $\infty - \infty \quad \frac{1}{x} - \frac{2}{x} \to -\infty \quad \frac{2}{x} - \frac{1}{x} \to \infty \quad$ as $x \to 0^+$

d) $0 \cdot \infty \quad (\sin x) \cdot (\frac{1}{x}) \to 1$, $\quad (\sin x)(\frac{1}{\sqrt{x}}) \to 0 \quad \Big\}$ as $x \to 0^+$

e) $0^0 \quad \lim_{x \to 0^+} 0^x = 0$

$\lim_{x \to 0^+} x^0 = 1$

harder. f) 1^∞ (hint: PSet 9, A6). (many other approaches, too).

$\lim_{x \to \infty} 1^x = 1 \qquad \lim_{x \to 1^+} x^{(x/\ln x)} = \infty$

$\lim_{x \to 0} (1+x)^{1/x} = e$

Example 0/0 is indeterminate since

$$\lim_{x \to 0} \frac{2x}{x} = 2 \quad \text{but} \quad \lim_{x \to 0} \frac{x}{2x} = \frac{1}{2}$$

even though both have the form $\frac{0}{0}$ if evaluated naively.

More precisely: if $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then

$\lim_{x \to c} \frac{f(x)}{g(x)}$ could still be anything.

② Evaluate the following limits. (We will probably return to these on Thursday).

~~a) $\lim_{x\to\infty} (x(\tan^{-1}x$~~

a) $\lim_{x\to\infty}\left[x\cdot\left(\tan^{-1}x-\frac{\pi}{2}\right)\right] = \lim \frac{\tan^{-1}x-\pi/2}{1/x} = \lim \frac{\frac{1}{1+x^2}}{-1/x^2} = \lim -\frac{x^2}{1+x^2}$

$\infty\cdot 0$ $\qquad \frac{0}{0}$ $\qquad = \boxed{-1}.$

b) $\lim_{x\to 0^+}(x^x) = \lim e^{x\ln x} = e^{\lim x\ln x} = e^{\lim \frac{\ln x}{1/x}} = e^{\lim \frac{1/x}{-1/x^2}}$

0^0 $\qquad = e^{\lim(-x)} = \boxed{1.}$

c) $\lim_{x\to 1^+}\left(x^{1/(x-1)}\right) = e^{\lim \frac{\ln x}{x-1}} = e^{\lim \frac{1/x}{1}} = e^1 = \boxed{e}$

1^∞

d) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) = \lim \frac{x}{\sqrt{x^2+x}+x} = \lim \frac{1}{\sqrt{1+1/x}+1} = \boxed{\frac{1}{2}}$

<u>Note</u> One of these is much easier to do <u>without</u> l'hôpital's rule.