Worksheet for 26 November

① $\int_0^2 x\cdot 2^x\,dx = \left[x\cdot\frac{1}{\ln 2}\cdot 2^x\right]_0^2 - \int_0^2 \frac{1}{\ln 2}\cdot 2^x\,dx$

$u=x \quad dv=2^x dx$
$du=dx \quad v=\frac{1}{\ln 2}\cdot 2^x$

$= 2\cdot\frac{1}{\ln 2}\cdot 4 - 0 - \left[\frac{1}{(\ln 2)^2}\cdot 2^x\right]_0^2$

$= \frac{8}{\ln 2} - \left(\frac{1}{(\ln 2)^2}\cdot 4 - \frac{1}{(\ln 2)^2}\cdot 1\right)$

$= \boxed{\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2}}$

② $\int x\cdot\sin(2x)\,dx = -\frac{1}{2}x\cos(2x) - \int\left(-\frac{1}{2}\right)\cos(2x)\,dx$

$u=x \quad dv=\sin(2x)dx$
$du=dx \quad v=-\frac{1}{2}\cos(2x)$

$= \boxed{-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C}$

③ $\int \ln x\,dx = (\ln x)\cdot x - \int x\cdot\frac{1}{x}\,dx$

$u=\ln x \quad dv=dx$
$du=\frac{1}{x}dx \quad v=x$

$= x\ln x - \int dx$

$= \boxed{x\ln x - x + C}$

④ $\int x^2 e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \int\frac{1}{2}\cdot 2x\cdot e^{2x}\,dx$

$u=x^2 \quad dv=e^{2x}dx$
$du=2x\,dx \quad v=\frac{1}{2}e^{2x}$

1st use of parts

$= \frac{1}{2}x^2e^{2x} - \int x\cdot e^{2x}\,dx$

$\boxed{u=x \quad dv=e^{2x}dx \quad du=dx \quad v=\frac{1}{2}e^{2x}}$

2nd use of parts

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}x\cdot e^{2x} + \int\frac{1}{2}e^{2x}\,dx$

$= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}x\cdot e^{2x} + \frac{1}{4}e^{2x} + C}$

The remaining problems require a combination of multiple techiques.

(5) $\int_0^1 \arctan x \, dx$

$u = \arctan x \quad dv = dx$

$du = \frac{1}{1+x^2}dx \quad v = x$

a) int. by parts

$= \left[x \cdot \arctan x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$ $\quad w = 1+x^2$, $dw = 2x\,dx$

b) subst.

$= 1 \cdot \frac{\pi}{4} - 0 \cdot 0 - \int_1^2 \frac{1/2}{u} du$

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln u\right]_1^2$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$

(6) $\int \cos(\sqrt{x})dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

a) substitute

$= \int \cos(u) \cdot 2\sqrt{x}\, du = \int 2u \cdot \cos(u) du$

b) parts

$v = 2u \quad dw = \cos(u)du$

$dv = 2du \quad w = \sin u$

$= 2u \sin u - \int 2 \sin u\, du = 2u \sin u + 2\cos u + C$

$= \boxed{2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C}$

(7) $\int (\ln x)^2 dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$ $\quad$ a) substitute; note $dx = x \cdot du = e^u \cdot du$

$= \int u^2 \cdot e^u du$ $\quad v = u^2 \quad dw = e^u du$, $dv = 2u\,du \quad w = e^u$ $\quad$ b) parts

$= u^2 e^u - \int 2u e^u du$ $\quad v = 2u \quad dw = e^u du$, $dv = 2du \quad w = e^u$ $\quad$ c) parts again

$= u^2 e^u - 2u \cdot e^u + \int 2e^u du$

$= \boxed{u^2 e^u - 2u \cdot e^u + 2e^u + C} = \boxed{(\ln x)^2 \cdot x - 2(\ln x) \cdot x + 2x + C}$

(8) $\int x^3 \cdot e^{-x^2/2} dx$

$u = -x^2/2$

$du = -x\,dx$

a) substitute

$= \int x^3 \cdot e^u \cdot \frac{du}{(-x)}$

$= \int (-x^2) \cdot e^u du$

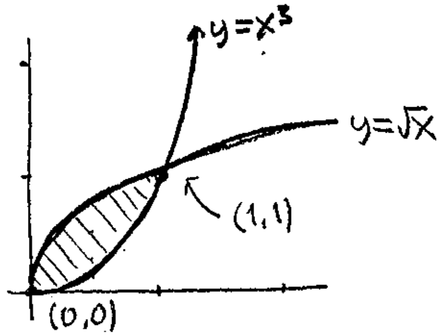
$= \int 2u \cdot e^u du$ $\quad v = 2u \quad dw = e^u du$, $dv = 2du \quad w = e^u$ $\quad$ b) parts

$= 2u \cdot e^u - \int 2e^u du = 2u \cdot e^u - 2e^u + C$

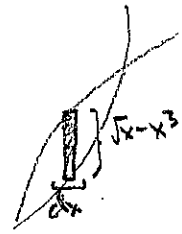
$= \boxed{-x^2 \cdot e^{-x^2/2} - 2e^{-x^2/2} + C}$

Part 2

① Find the area between the curves $y = x^3$ and $y = \sqrt{x}$:



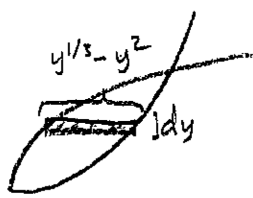
a) By slicing vertically



$$\int_0^1 (\sqrt{x} - x^3)\,dx$$
$$= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^4\right]_0^1$$
$$= \tfrac{2}{3} - \tfrac{1}{4} = \boxed{\tfrac{5}{12}}$$

b) By slicing horizontally.

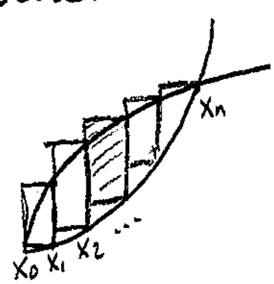
$y = \sqrt{x} \iff x = y^2$
$y = x^3 \iff x = y^{1/3}$



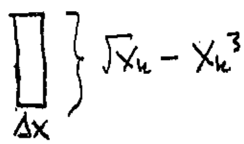
$$\int_0^1 (y^{1/3} - y^2)\,dy = \left[\tfrac{3}{4}y^{4/3} - \tfrac{1}{3}y^{1/3}\right]_0^1$$
$$= \tfrac{3}{4} - \tfrac{1}{3} = \boxed{\tfrac{5}{12}}$$

c) Write Riemann sum approximations (n rectangles) for both cases.

Vertical:

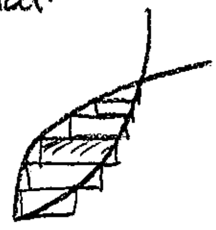


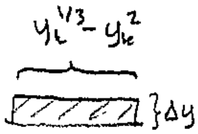
area of k^{th} rectangle :



so :
$$\boxed{\text{area} \approx \sum_{k=1}^{n} (\sqrt{x_k} - x_k^3)\cdot\Delta x \quad \text{where } \Delta x = 1/n,\; x_k = k/n.}$$

Horizontal:

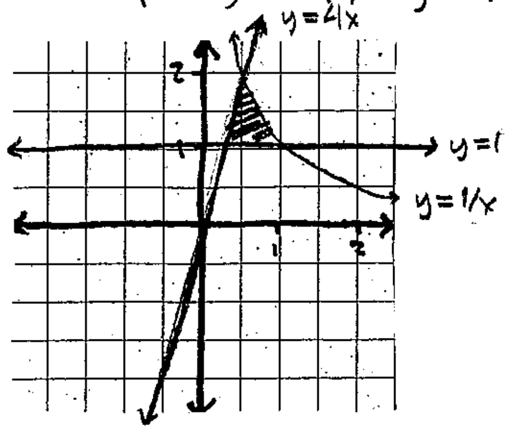




$$\text{area} \approx \sum_{k=1}^{n} (y_k^{1/3} - y_k^2)\cdot\Delta y$$

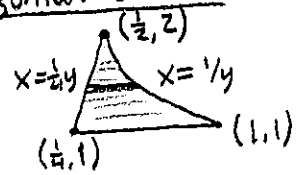
$\Delta y = 1/n$
$y_k = k/n$

② a) Graph $y=4x$, $y=1/x$, and $y=1$ on the axes below.



b) Compute the area of the region bounded by these three curves, by slicing vertically or horizontally.

Horizontal is easier:

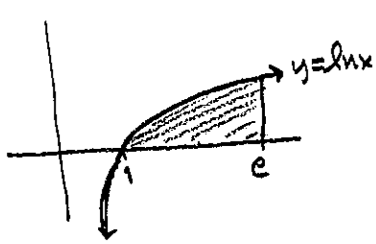


$$\int_1^2 \left(\frac{1}{y}-\frac{1}{4}y\right)dy = \left[\ln|y| - \frac{1}{8}y^2\right]_1^2$$
$$= \left(\ln 2 - \frac{1}{8}\cdot 4\right) - \left(\cancel{\ln 1}^{\,0} - \frac{1}{8}\cdot 1\right)$$
$$= \boxed{\ln 2 - 3/8}$$

Vertical is also possible:

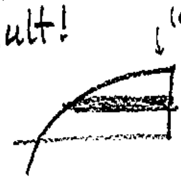
$$\text{area} = \int_{1/4}^{1/2}(4x-1)\,dx + \int_{1/2}^{1}\left(\frac{1}{x}-1\right)dx$$
$$= \left[2x^2-x\right]_{1/4}^{1/2} + \left[\ln|x|-x\right]_{1/2}^{1} = \left(2\cdot\frac{1}{4}-\frac{1}{2}\right)-\left(2\cdot\frac{1}{16}-\frac{1}{4}\right)+(0-1)-\left(\ln\frac{1}{2}-\frac{1}{2}\right)$$
$$= \cancel{\frac{1}{2}-\frac{1}{2}} - \frac{1}{8}+\frac{1}{4}-1-\ln\frac{1}{2}+\frac{1}{2}$$
$$= \boxed{\ln 2 - 3/8}$$

③ a) Compute $\int_1^e \ln x\,dx$ using integration by parts.



$$= \left[x\ln x - x\right]_1^e = (e\cdot 1 - e) - (1\cdot 0 - 1)$$ (using problem #3)
$$= \boxed{1}$$

b) Compute the same area by slicing horizontally; make sure you get the same result!



$$\int_0^1 (e-e^y)\,dy = \left[e\cdot y - e^y\right]_0^1$$
$$= (e\cdot 1 - e) - (e\cdot 0 - 1)$$
$$= \boxed{1}$$