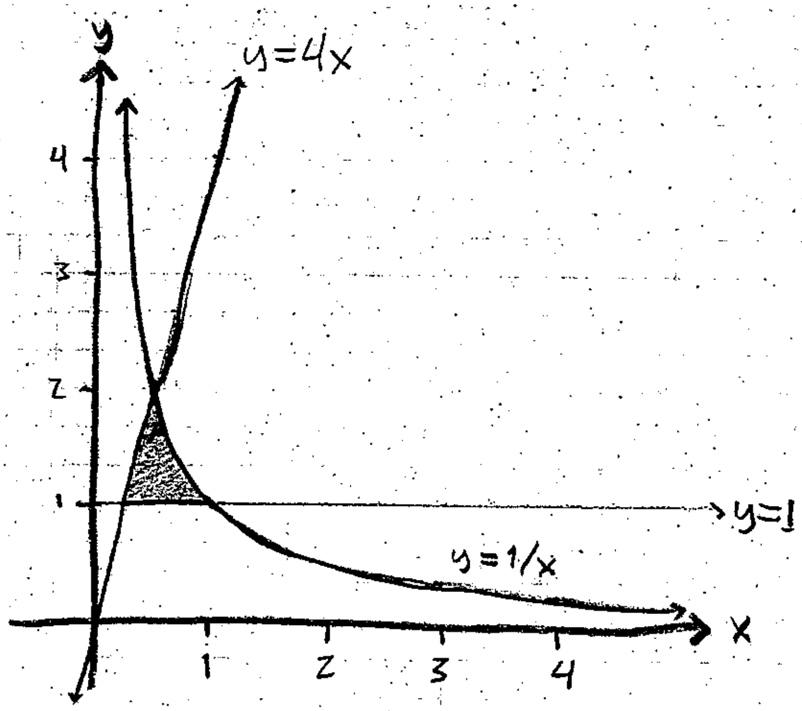
Worksheet for 12/3/13

Part 1

(1)

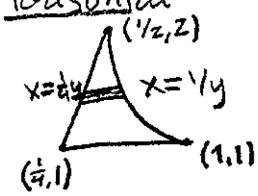


Compute the area of the region bounded by

$y=1$,
$y=4x$, and
$y=1/x$,

by vertical or horizontal slicing.

Horizontal:



$$\int_1^2 \left(\frac{1}{y} - \frac{1}{4}y\right) dy$$

$$= \left[\ln|y| - \frac{1}{8}y^2\right]_1^2$$

$$= \ln 2 - \frac{1}{8}\cdot 4 - \ln 1 + \frac{1}{8}$$

$$= \boxed{\ln 2 - \frac{3}{8}} \approx 0.318$$

Vertical:

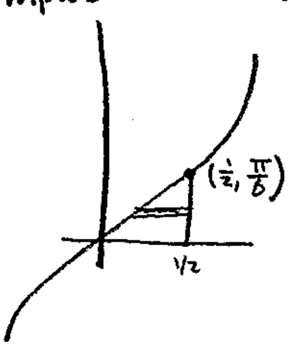
$$\int_{1/4}^{1/2} (4x-1)\,dx + \int_{1/2}^{1} \left(\frac{1}{x} - 1\right) dx$$

$$= \left[2x^2 - x\right]_{1/4}^{1/2} + \left[\ln x - x\right]_{1/2}^{1}$$

$$= 2\cdot\frac{1}{4} - \frac{1}{2} - 2\cdot\frac{1}{16} + \frac{1}{4} + \ln 0 - 1 - \ln\frac{1}{2} +$$

$$= \boxed{\ln 2 - \frac{3}{8}}$$

(2) Compute $\int_0^{1/2} \arcsin x\,dx$ by horizontal slicing. (It is also possible to compute this using integration by parts).



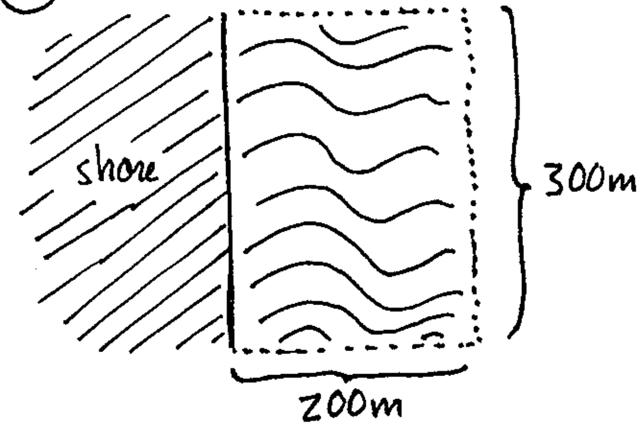
$$\int_0^{\pi/6} \left(\frac{1}{2} - \sin y\right) dy = \left[\frac{1}{2}y + \cos y\right]_0^{\pi/6}$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 0 - 1$$

$$= \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1}$$

Part 2

①



A certain patch of ocean is home to a population of jellyfish. The density of the jellyfish depends on the distance to the shore: it is

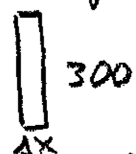
$\rho(x)$ jellyfish per m^2

where $x=$ distance to shore (in meters).

We want to calculate the number of jellyfish in the 200m by 300m region shown.

a) How could you slice this region into n rectangles so that the jellyfish density is close to constant in each rectangle?

n vertical rectangles, each 300m by $\left(\frac{200}{n}\right)$ meters.



b) What is the approximate number of jellyfish in the k^{th} slice?

$$300\cdot\Delta x\cdot\rho(x_k) \quad \text{where} \quad \Delta x=\frac{200}{n}$$

c) Write a sum that gives the approximate total number of jellyfish.

$$\sum_{k=1}^{n} 300\rho(x_k)\cdot\Delta x \quad \text{where} \quad \begin{array}{l} x_k = k\cdot\Delta x \\ \Delta x = 200/n \end{array}$$

d) Write an integral to compute the number of jelly fish.

$$\int_0^{200} 300\rho(x)\,dx$$

e) Compute this integral in the case that $\rho(x)=2^{-x/50}$.

$$\int_0^{200} 300\cdot 2^{-x/50}dx = \int_0^{-4} 300\cdot 2^{u}(-50)du = \left[\frac{-15000}{\ln 2}\cdot 2^{u}\right]_0^{-4}$$

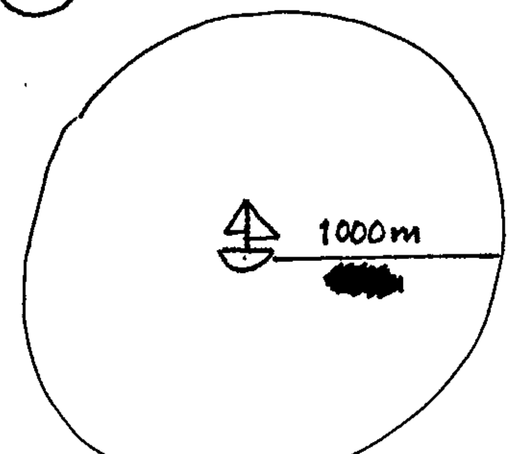
$$= \boxed{\frac{15000}{\ln 2}\left(1-2^{-4}\right)} = \frac{28125}{\ln 2}$$

$$\approx 20{,}288.$$

$u = -x/50$

$du = -1/50$

②



Some sharks are forming a group around your boat. Suppose there are $\rho(r)$ sharks per square meter at distance r meters from your boat.

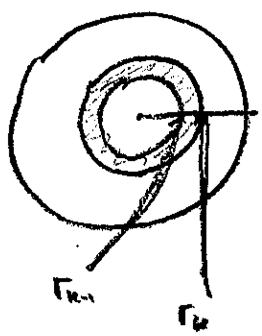
a) Write an integral that gives the total number of sharks within 1000m of your boat.

(follow the same basic steps as parts (a)-(d) of problem 1, but this time do not slice into rectangles. How should you slice?)

Slice into "washers". Let $\Delta r = 1000/n$; $r_k = k \cdot \Delta r$

washer area $\approx 2\pi \cdot r_k \cdot \Delta r$

density $\approx \rho(r_k)$



$$\#\text{sharks} \approx \sum_{k=1}^{n} 2\pi r_k \rho(r_k) \Delta r$$

$$= \boxed{\int_0^{1000} 2\pi r \rho(r)\, dr} \quad \text{(passing to the limit)}$$

b) Compute this integral in case $\rho(r) = e^{-r^2/1000}$

$$\int_0^{1000} 2\pi r e^{-r^2/1000}\, dr = \int_0^{10^6} \pi \cdot e^{-u/1000}\, du = \left[-1000\pi e^{-u/1000}\right]_0^{10^6}$$

$u = r^2$; $du = 2r\,dr$

$$= \boxed{1000\pi - 1000\pi \cdot e^{-1000}} \approx 3142.$$

c) Compute this integral in case $\rho(r) = e^{-r/1000}$

integrate by parts:

$$\int_0^{1000} 2\pi r e^{-r/1000}\, dr = \left[-2000\pi r e^{-r/1000}\right]_0^{1000} + \int_0^{1000} 2000\pi \cdot e^{-r/1000}\, dr$$

$u = 2\pi r$ $dv = e^{-r/1000} dr$

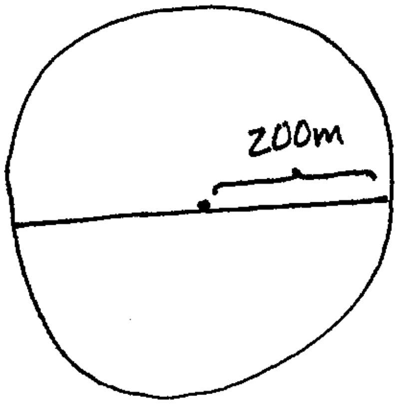
$du = 2\pi dr$ $v = -1000 e^{-r/1000}$

$$= -2000\pi \cdot 1000/e + \left[2000\pi \cdot 1000 \cdot e^{-r/1000}\right]_0^{1000}$$

$$= -2000\pi \cdot 1000/e - 2000\pi \cdot 1000/e + 2000\pi \cdot 1000 = \boxed{2000000\pi \cdot (1 - \tfrac{2}{e})}$$

1.6 million !!

(3)

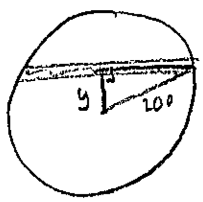


A circular lake of radius 200m has a rope across the middle. There are $\rho(y)$ lily-pads per square meter at a distance y meters from this rope.

a) Write an integral that gives the number of lily pads in the lake.

Top half: $\displaystyle\int_0^{200} 2\sqrt{200^2-y^2}\cdot\rho(y)\,dy$

bottom half is equal to the top (by symmetry).



height $= \Delta y$
width $= 2\sqrt{200^2-y^2}$

Thus:

total $= \boxed{\displaystyle\int_0^{200} 4\sqrt{200^2-y^2}\,\rho(y)\,dy.}$