Lecture 13: Optimization I

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1 Introduction

In this lecture, we apply the techniques from lecture 11 (maximizing and minimizing functions) to solve so-called optimization problems. Essentially, this means we will be looking at word problems that can be solved by maximizing or minimizing some function. These problems are solved in two stages: first you must reformulate them as maximizing or minimizing some function, and second you must solve this maximization or minimization problem (using the techniques we saw last week).

The reference for today is Stewart §4.6.

2 The basic strategy

I will first illustrate the sort of optimization problems we will solve with a relatively simple example, and then state the basic strategy shown in this example.

Example 2.1. Suppose that a farmer wishes to construct a rectangular chicken pen using exactly 8 feet of fence. What is the largest area that such a pen could enclose?

Solution. First, introduce two relevant variables: call h and w the two dimensions of the rectangular pen (for height and width). So the area of the pen is hw. Now, h and w are related by an equation: the total perimeter of the pen must be 8, and the pen has two sides of length w and two sides of length h. Therefore $2w + 2h = 8$; in other words $w + h = 4$. In fact, you can use this constraint to solve for one variable in terms of the other: $h = 4 - w$. So the area of the pen is $(4 - w)w$. Note that w must be chosen between 0 and 4, since neither side (h or $4 - h$) can have negative length. So the area of the pen must be $(4 - w)w$, where w is chosen from the interval $[0, 4]$. So we can use the closed interval method to find the maximum value of the function $A(w) = (4 - w)w$ on the interval $[0, 4]$. The derivative is $A'(w) = 4 - 2w$, which is always defined, and $A'(w) = 0$ if and only if $4 = 2w$, i.e. $w = 2$. So there is a unique critical number of $A(w)$, namely 2. Checking this critical number and the two endpoints gives $A(0) = 0, A(2) = 4, A(4) = 2$. So the maximum of these is the absolute maximum: $A(2) = 4$. Since the area of any chicken pen must be some value of $A(w)$, it follows that the area of the pen cannot be larger than 4. On the other hand, taking $w = 2$ (and therefore $h = 4 - w = 2$) gives a pen of the maximum area, which is a square pen with sides of length 2 and area $\boxed{4}$.

This example shows the basic strategy that we will always employ in these sorts of problems, which is as follows.

1. Introduce variables for any relevant quantities in the problem. Write the quantity to be optimized in terms of these variables.

2. Write down any constraints that must exist between the variables. Use this to solve for some of the variables in terms of the others.

3. Write the quantity to be optimized as a function of a single variable (using the result of the previous step), and determine the interval of possible inputs to this function.

4. Find the maximum (or minimum) of this function on the interval you have determined, using methods from lecture 11.

Note. There are many tricks that can be used to make these sorts of problems easier. Here are a couple; we'll see these illustrated in examples later.

- If there are several variables (related by constraints), try to work in terms of the one variables which will make your work easiest (in the example above, it doesn't matter if you work in terms of h or w, but we'll see other examples where one choice is much easier than another).
- Sometimes, if you want to optimize one quantity, it is easier to optimize its square (or some other function of it). For example, it is usually easier to optimize the square of a distance than the distance itself (we'll see an example of this in the next section).

3 Examples

Here are a few more example problems, which can be solved using the differentiation techniques currently at our disposal. In lecture 15 we'll see some more examples using more differentiation techniques.

Example 3.1. Suppose now that the farmer wants to enclose exactly 10 square feet, and want to use the minimum possible amount of fence. How much fence does he need? As before, he'll be using a rectangular pen.
Solution. As before, let h, w be the two dimensions of the pen. What makes this problem different from before is that the constraint and the quantity to be optimized have switched places. The constraint is now $hw = 10$ (the area *must* be 10), while the quantity to be optimized is the perimeter $2w + 2h$. Now use the constraint to solve for one variable: for example, we can write $w = 10/h$. Then the perimeter is now $P(h) = 2 \cdot \frac{10}{h} + 2h = \frac{20}{h} + 2h$. In this case, h could be anything in the interval $(0, \infty)$ (of course practically speaking you would never make h anywhere near 0 or ∞, but it turns out now to matter for the optimization). Then $P'(h) = -\frac{20}{h^2} + 2$, which is defined for all positive h. So $P'(h) = 0$ if and only if $2 = \frac{20}{h^2}$, i.e. $h = \sqrt{10}$. So the function $P(h)$ has a unique critical number, namely $\sqrt{10}$. Now, $P''(w) = \frac{40}{h^3}$, which is always positive (when h is positive), so $P(h)$ is concave up on $(0, \infty)$; it follows that $P(h)$ has a local minimum at $h = \sqrt{10}$, and that this is also a global minimum (because the whole graph must lie above the tangent line at $h = \sqrt{10}$, which is horizontal). So the minimum possible amount of fence needed is $P(\sqrt{10}) = \frac{20}{\sqrt{10}} + 2\sqrt{10} = \boxed{4\sqrt{10}}$. This minimum is achieved by taking $h = \sqrt{10}$ and $w = \frac{10}{\sqrt{10}} = \sqrt{10}$; this is, the shape of the pen should be a square.

Example 3.2. Suppose that you are constructing an aluminum can that must hole $500mL$ (i.e. its volume must be $500mL$). The can is shaped like a cylinder. So the cost of making this can is given by the surface area (in cm^2) of this cylinder times the price of aluminum per square centimeter. Therefore you'd like to design the can to have the minimum possible surface area. What is this minimum possible surface area?
Solution. Begin by introducing the following variables: let r be the radius of the can, and h the height. Then you an express the volume and surface area as follows[1]

$$\begin{aligned} \text{Volume} &= \pi r^2 h \\ \text{Surface area} &= 2\pi r^2 + 2\pi r h \end{aligned}$$

Let's assume that r and h are measured in cm, so that surface area is measured in cm^2 and volume is measured in cm^3 (or mL, which is the same thing).

[1] We certainly don't expect you to know this surface area formula from memory, but it isn't too difficult to obtain: just divide the surface area into the two circles on top and bottom, each of area πr^2, and the side, which has circumference $2\pi r$ and height h, for area $2\pi rh$.

The constraint we know is that the volume of the can must be $500mL$. That is, $\pi r^2 h = 500$. We should solve for one of the variables. In this case, it is somewhat easier to solve for h, since we can avoid taking a square root in this case: $h = \frac{500}{\pi r^2}$. Then the volume can be expressed as the following function of r.

$$V(r) = 2\pi r^2 + \frac{1000}{r}$$

Here, r must be chosen to be positive, i.e. from the interval $(0, \infty)$. We can find the first two derivatives of V using the power rule.

$$\begin{aligned} V'(r) &= 4\pi r - \frac{1000}{r^2} \\ V''(r) &= 4\pi + \frac{2000}{r^3} \end{aligned}$$

From the expression for $V''(r)$, we see that the function is always concave up on the interval $(0, \infty)$, meaning that any critical number gives a local minimum, which must in fact be a absolute minimum as well. To find the critical value, solve the equation $0 = V'(r)$, to obtain $4\pi r^3 = 1000$, or $r = \sqrt[3]{\frac{500}{\pi}}$. This is the unique critical value, so it gives the unique absolute minimum, which is $V(r) = 2\pi \left(\frac{500}{\pi}\right)^{2/3} + \frac{1000}{\sqrt[3]{500/\pi}}$. This could be simplified further, but the simplification is not too illuminating: the approximate value is 349. So the smallest possible surface area is $\boxed{\approx 349cm^2}$, which is achieved for $r \approx 4.3cm$ and $h \approx 8.6cm$.

By the way, one way to phrase the upshot of all the analysis above is: to minimize the surface area of the can, you should choose the diameter of the base to be equal to the height. This is approximately true for a standard 1 gallon paint can, which is $6\frac{1}{2}$ inches in diameter and $7\frac{3}{4}$ inches in height. Of course (as you can readily see by thinking of other cans you've seen) other proportions are also common, reflecting the fact that surface area is not always the most important feature to optimize. The optimal choice will depend on many other factors.

Example 3.3. Find the minimal distance from the point $(3, 4)$ to the line described by $4x + 3y = 12$.
Note. There are of course much better ways to solve this problem than using calculus; in particular it can be solved by a bit of middle school geometry. But the problem here serves as an illustration of the techniques we're using; in particular, we will ask slightly different problems on the homework, where the same technique described here (using calculus) will work, whereas an elementary calculation would be much more difficult. In any case, if you can find the elementary method, I certainly encourage you to carry it out and then check to make sure that it gives the same answer[2].
Solution. Introduce variables x and y; let them be the coordinates of some point on the line. These are related by the constraint $4x + 3y = 12$, which you can solve to obtain $y = 4 - \frac{4}{3}x$. The quantity we wish to optimize is the distance d from (x, y) to $(3, 4)$. This can be computed by the standard distance formula between two points in the plane (also known as the Pythagorean theorem).

[2]In fact, I solved this problem first with an elementary method, as a way of checking that I did the calculus correctly.

$$
\begin{aligned}
d &= \sqrt{(3-x)^2 + (4-y)^2} \\
&= \sqrt{9 - 6x + x^2 + (4 - (4 - \frac{4}{3}x)^2)} \\
&= \sqrt{9 - 6x + x^2 + \left(\frac{4}{3}x\right)^2} \\
&= \sqrt{9 - 6x + \left(1 + \frac{16}{9}\right) x^2} \\
&= \sqrt{9 - 6x + \frac{25}{9}x^2}
\end{aligned}
$$

At this point we are a little stuck with our current methods. However, a basic insight makes the problem soluble (and even if we had more advanced methods, this trick would make the solution somewhat cleaner): to find the minimum value of d, it suffices to just find the minimum value of d^2 (since d is positive). And that will be much easier, since d^2 is just a polynomial in x. So let

$$f(x) = 9 - 6x + \frac{25}{9}x^2$$

and notice that $d^2 = f(x)$. So now we just need to minimize $f(x)$, where x ranges over $(-\infty, \infty)$. But this is by now fairly routine.

$$
\begin{aligned}
f'(x) &= -6 + \frac{50}{9}x \\
f''(x) &= \frac{50}{9}
\end{aligned}
$$

Since $f''(x)$ is always positive, f is concave up, so any local minimum is an absolute minimum. The only critical value is the solution of $f'(x) = 0$, i.e. $\frac{50}{9}x = 6$; this solution is $x = \frac{27}{25}$. This minimum function value is $f(\frac{27}{25}) = 9 - 6 \cdot \frac{27}{25} + \frac{25}{9} \cdot \frac{27^2}{25^2} = \frac{144}{25}$.

Therefore, the minimum value of d^2 is $\frac{144}{25}$; it follows that the minimum value of d is $\sqrt{\frac{144}{25}} = \frac{12}{5} = \boxed{2.4}$. This is the minimum distance from $(3, 4)$ to the given line.