Lecture 15: Optimization II

Nathan Pflueger

11 October 2013

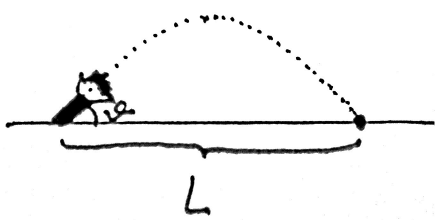
1 Introduction

There is no new material in this lecture. We will just do another collection of optimization problems, this time using the new differentiation techniques covered last time.

As with lecture 13, the reference is Stewart §4.6.

2 Examples

Example 2.1. If a cannonball is launched from the ground at an initial speed of v_0 meters per second, at an angle θ with the ground, the total distance it travels before hitting the ground is given by the following formula (you are not responsible to know this formula, but it is a good exercise to derive it if you know a little physics). Here g is the constant 9.8 meters per second per second (m/s^2), which measures the acceleration due to gravity on the surface of the earth.



$$L = \frac{2v_0^2}{g} \sin\theta \cos\theta$$

What launch angle θ causes the cannon ball to fly the furthest possible distance? Does it matter what the value of g or v_0 is (for example, is the best launch angle the same on the moon)?

Solution. Regard v_0, g as constants, and think of the total distance as a function $L(\theta)$ of θ. Then θ must be chosen in the interval $[0, \pi/2]$ (the two endpoints being shooting the cannon straight forward or shooting it straight up into the air). So let's maximize this function on the interval $[0, \pi/2]$. Begin by finding $L'(\theta)$.

$$\begin{aligned}
L(\theta) &= \frac{2v_0^2}{g} \sin\theta \cos\theta \\
L'(\theta) &= \frac{2v_0^2}{g} \left[(\sin\theta)' \cos\theta + \sin\theta (\cos(\theta))'\right] \text{ (Product rule)} \\
&= \frac{2v_0^2}{g} \left[\cos\theta \cos\theta - \sin\theta \sin\theta\right] \text{ (Derivatives of sine and cosine)} \\
&= \frac{2v_0^2}{g} \left(\cos^2\theta - \sin^2\theta\right)
\end{aligned}$$

Now solve $L'(\theta) = 0$ to find the critical numbers.

$$\begin{aligned} 0 &= \frac{2v_0^2}{g}\left(\cos^2\theta - \sin^2\theta\right) \\ \Leftrightarrow 0 &= \cos^2\theta - \sin^2\theta \\ \Leftrightarrow 0 &= (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) \\ \Leftrightarrow \cos\theta &= \pm\sin\theta \end{aligned}$$

Now, on the interval $[0, \frac{\pi}{2}]$, both $\sin\theta$ and $\cos\theta$ are positive, so $\cos\theta = \pm\sin\theta$ means that $\cos\theta = \sin\theta$, which occurs only at the angle $\theta = \frac{\pi}{4}$ (45 degrees). So this is the only critical number.

Evaluating the function at the critical number and endpoints gives:

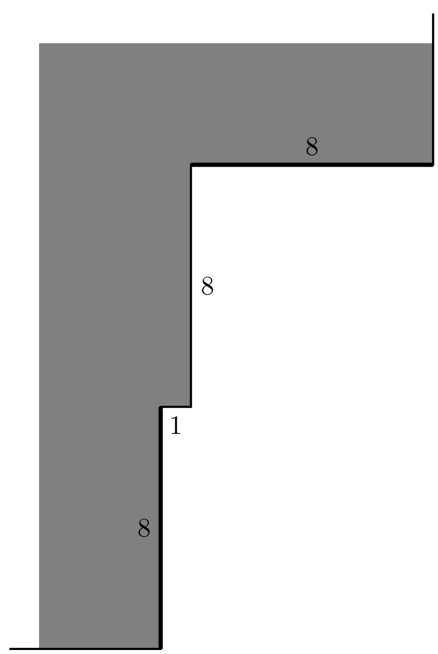
$$\begin{aligned} L(0) &= 0 \\ L(\pi/4) &= \frac{2v_0^2}{g}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = v_0^2/g \\ L(\pi/2) &= 0 \end{aligned}$$

So the absolute maximum occurs at the critical number $\boxed{\theta = \pi/4.}$ So the optimal launch angle is 45 degrees.

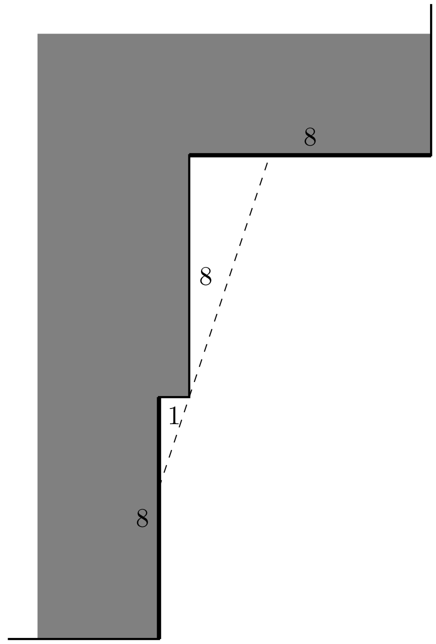
Notice that the optimal launch angle θ **does not depend on the initial velocity, or the strength of gravity.** The furthest possible distance does depend on these, but the launch angle does not. As you would expect, weaker gravity makes the cannonball fly further, as does a faster muzzle velocity.

Note. If you happen to know the double-angle formula for $\sin(2\theta)$, this problem can be solved somewhat more easily. We are not assuming knowledge of this formula however (at least not for now).

Example 2.2. Suppose that the footprint of the corner of a building looks like this.

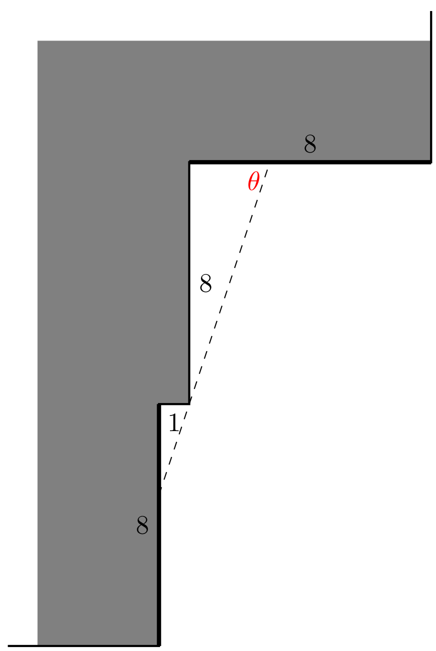


Suppose that a perfectly straight pipe must be placed, connecting the two bolded walls but not hitting the building, as shown with the dotted line.

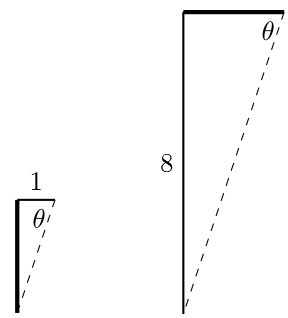


What is the shortest possible length that such a pipe could have?

Solution. Notice that the shortest possible pipe should definitely touch the "inner corner" of the building (as shown in the picture). Now introduce a variable angle θ, which gives the angle that the pipe makes with one wall (it doesn't matter which wall you choose).



Now the length of the pipe can be viewed as a function $L(\theta)$, whose value can be broken into two pieces. One piece comes from each of the two right-angled triangles in the picture.



$$L(\theta) \quad = \quad \sec\theta \quad + \quad 8\csc\theta$$

The way I have obtained this formula is: each of the two parts of the dashed line forms the hypotenuse of a right triangle in the picture. Both right triangles have the same angle θ in their corner. In one case the adjacent length (1) is known; in the other case the opposite length (8) is known. To convert from these lengths to the length of the hypotenuse requires the use of one of the elementary trigonometric functions (either secant or cosecant).

Once this formula is obtained, it is just a matter of finding the optimal value of θ. To do this, begin by computing the derivative $L'(\theta)$. As we saw in lecture 14, the functions secant and cosecant can be differentiated using the quotient rule; the result is the following.

$$L'(\theta) = \sec\theta\tan\theta - 8\csc\theta\cot\theta$$

To find the critical numbers, we must solve $L'(\theta) = 0$. This requires some care, but it is tractable. Here is one path to solving it: re-express the trig functions involved in terms of sine and cosine, simplify as much as possible and then re-express the equation again.

$$\begin{aligned}
0 &= \sec\theta\tan\theta - 8\csc\theta\cot\theta \\
8\csc\theta\cot\theta &= \sec\theta\tan\theta \\
8\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} &= \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} \\
8 &= \frac{\sin^3\theta}{\cos^3\theta} \\
8 &= \tan^3\theta \\
2 &= \tan\theta \\
\tan^{-1}(2) &= \theta
\end{aligned}$$

This value of θ, this inverse tangent of 2, does not have a simpler form. You can compute with a calculator that it is approximately 1.107 radians, or 63 degrees.

So the only critical number is $\theta = \tan^{-1}(2)$. To see what kind of extremum it is, look at the second derivative of $L(\theta)$.

$$\begin{aligned}
L''(\theta) &= (\sec\theta)'\tan\theta + \sec\theta(\tan\theta)' - 8(\csc\theta)'\cot\theta - 8\csc\theta(\cot\theta)' \\
&= \sec\theta\tan^2\theta + \sec^3\theta + 8\csc\theta\cot^2\theta + 8\csc^3\theta
\end{aligned}$$

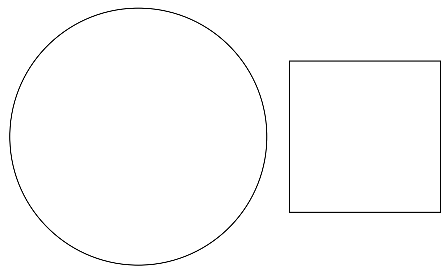
This is a pretty complicated expression, but fortunately we don't need to look closely at it: all of the terms are positive functions for any angle in $[0, \frac{\pi}{2}]$ (and any angle that is physically possible in this problem must be acute). So we can conclude that $L''(\theta) > 0$. This means that the function is concave up everywhere on the domain we care about, hence the critical number gives a local minimum and also a global minimum.

Therefore the minimum possible length of pope is given by the following expression.

$$L(\tan^{-1}(2)) = \sec(\tan^{-1}(2)) + 8\csc(\tan^{-1}(2))$$

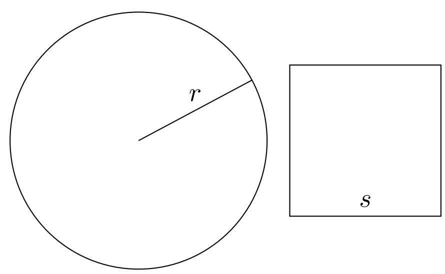
You can compute with a calculator that this value is approximately $\boxed{11.2 \text{ meters.}}$ Alternatively, you can draw a right triangle with legs of length 1 and 2 to see that $\sec(\tan^{-1}(2)) = \sqrt{5}$ and $\csc(\tan^{-1}(2)) = \sqrt{5}/2$. Therefore $L(\tan^{-1}(2)) = \sqrt{5} + 8 \cdot \sqrt{5}/2 = \boxed{5\sqrt{5}.}$

Example 2.3. Suppose that you clip a single piece of wire, $10cm$ long, into two pieces. One piece you bend into a circle, and the other you bend into a square.



Consider the total area enclosed by the circle and the square together. What is the **maximum** total area that can be enclosed in this way?

Solution. First introduce two new variables r and s to denote the radius of the circle and the side length of the square, in centimeters.



Then the total area is given by $\pi r^2 + s^2$, and the constraint that the original wire was 10 centimeters long can be expressed by $2\pi r + 4s = 10$ (this just says that the perimeters of the two figures add up to 10). Solve for one variable to obtain $s = \frac{5-\pi s}{2}$. Now substitute this back to obtain the area as a function of the other variable, r.

$$\begin{aligned} A(r) &= \pi r^2 + \left(\frac{5-\pi r}{2}\right)^2 \\ &= \left(\pi + \frac{1}{4}\pi^2\right) r^2 - \frac{5\pi}{2} r + \frac{25}{4} \end{aligned}$$

The interval of possible values of r begins at 0 (this is the case where you don't clip the wire, and only make a square) and ends at $\frac{5}{\pi}$ (this is the case where you only make a circle of circumference 10). So the relevant interval is $[0, \frac{5}{\pi}]$.

In this case, we could take the derivative and find the critical numbers as usual. But in this particular case, we can actual skip the algebra. The reason is that the second derivative of this function is $A''(r) = 2\left(\pi + \frac{1}{4}\pi^2\right)$, which is positive. So this function is concave up everywhere. In particular, any critical numbers are local *minima*, not local maxima. So *the absolute maximum must occur at an endpoint.* To see which endpoint, just check the value of the function at the endpoints.

$$\begin{aligned} A(0) &= \frac{25}{4} \\ A(\frac{5}{\pi}) &= \frac{25}{\pi} \end{aligned}$$

The larger of the two of these is $\boxed{25/\pi \approx 7.96.}$ So this is maximum possible area: it is achieved when you turn the entire wire into a circle, and none of it into a square.

If you want to check you work: the critical number here is in fact $\frac{5}{4+\pi} \approx 0.61$, and it gives the absolute minimum, which is approximately 3.5. That is the *minimum* possible total area of the square and rectangle.

Example 2.4. Suppose that you are selling popcorn at a movie theater, for a price of p dollars per bag. It costs you 1 dollar per bag to make the popcorn. You are trying to work out what price you should set p to in order to maximize your profits. By trying out a few different prices, you observe the following facts.

- If the price is two dollars per bag, then 200 customers will buy a bag of popcorn.
- For every 25 cents you increase the price, 5 fewer people are willing to buy the popcorn.

What price p should you charge?

Solution. The basic equation here is the following.

$$\text{total profit} = (\text{profit per bag}) \cdot (\text{number sold})$$

The profit per bag is easy to write down in terms of p: you take in the p dollars the customer pays, but lose the 1 dollar it costs you to make the bag.

$$\text{profit per bag} = p - 1$$

The number of customers takes a bit of algebra to work out. You know that if $p = 2$ then the number sold is 200. You also know that the number sold appears to be linear: it increases by 5 when p increases by 0.25. So it increases by 20 when p increases by 1. In other words, it has slope 20 as a function of t. But we know how to find the equation of a linear function with slope 20 and which is equal to 200 at $p = 2$.

$$\begin{aligned} \text{number sold} &= 200 + 20 \cdot (p - 1) \\ &= 240 - 20p \end{aligned}$$

So putting these things together, we obtain the following function, which tells the total profit if the price is set at p dollars.

$$\begin{aligned} f(p) &= (p-1)(240 - 20p) \\ &= -20p^2 + 260p - 240 \end{aligned}$$

We are looking for the absolute maximum of this function. It is a downward facing quadratic function (since $f''(p) = -40$ is negative), so it is always concave down, and its local maximum is a global maximum. To find this maximum, notice that $f'(p) = -40p + 260$. Setting this equal to 0 gives $40p = 260$, or $\boxed{p = 6.5.}$ So the optimal price for a bag of popcorn is 6.5 dollars per bag. This price will result in a total profit of $f(6.5) = (6.5 - 1)(240 - 20 \cdot 6.5) = 5.5 \cdot 110 = 605$ dollars.

As someone said in class, this is a pretty ridiculous price for a bag of popcorn. But of course we're talking about a movie theater, where the captive audience creates a much different demand curve than you'd see elsewhere. Then there's the minor feature that I made all these numbers up. But the principle holds in real markets: the best price isn't the price everyone will pay, it's the price that perfectly balances the loss of willing customers with the increased profit per customer.

Example 2.5. Sketch the graph of the function $f(x) = \sin x + \sin^2 x$ on the interval $[0, 2\pi]$.

Solution. First, compute the derivative of this function. We can do this using the product rule (later, we'll also be able to use the "chain rule," but they're both equally fast).

$$\begin{aligned} f(x) &= \sin x + \sin x \cdot \sin x \\ f'(x) &= (\sin x)' + (\sin x)' \sin x + \sin x (\sin x)' \\ &= \cos x + 2 \sin x \cos x \\ &= \cos x (1 + 2 \sin x) \end{aligned}$$

To find the critical points, solve $0 = \cos x(1+2\sin x)$. This equation holds precisely when either $\cos x = 0$ or $1 + 2\sin x = 0$. That is, either $\cos x = 0$ or $\sin x = -\frac{1}{2}$. Solving these two equations separately gives the critical numbers $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Arranged in increasing order, these are $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

To determine which of these are which type of extremum, compute the second derivative.

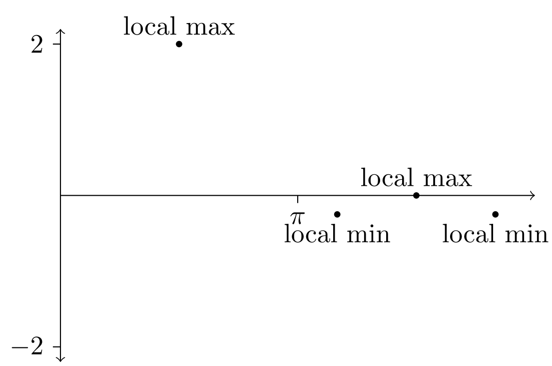
$$\begin{aligned} f''(x) &= -\sin x + 2 \cos x \cos x - 2 \sin x \sin x \\ &= -\sin x + 2 \cos^2 x - 2 \sin^2 x \end{aligned}$$

This will be easier to study if it is expressed using only sines, so substitute $\cos^2 x = 1 - \sin^2 x$ to obtain the following expression.

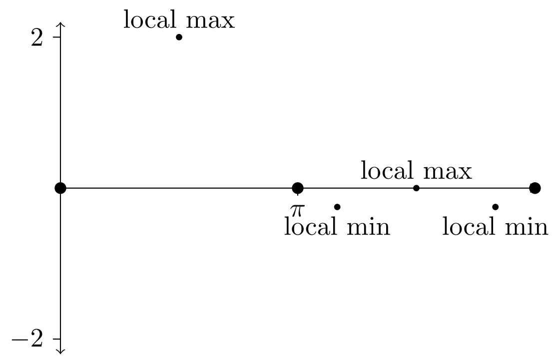
$$\begin{aligned} f''(x) &= -\sin x + 2 - 2\sin^2 x - 2 \sin^2 x \\ &= -4 \sin^2 x - \sin x + 2 \end{aligned}$$

Evaluating this at the four critical numbers gives $f''(\frac{\pi}{2}) = 2, f''(\frac{7\pi}{6}) = \frac{3}{2}, f''(\frac{3\pi}{2}) = -1, f''(\frac{11\pi}{6}) = 2$. Thus these are, respectively: a local max, local min, local max, local min.

Computing the value of the function at these critical points gives $f(\frac{\pi}{2}) = 2, f(\frac{7\pi}{6}) = -\frac{1}{4}, f(\frac{3\pi}{2}) = 0, f(\frac{11\pi}{6}) = -\frac{1}{4}$. So begin by plotting these four extrema.



Next, let's fine the x-intercepts. Notice that $f(x) = 0$ if and only if $\sin x + \sin^2 x = 0$, i.e. $\sin x(1+\sin x) = 0$, i.e. $\sin x = 0$ or $\sin x = -1$. This gives x values $0, \pi, \frac{3\pi}{2}$, and 2π. So add these points to the plot also.



These are enough points to get a pretty good idea of the curve's shape. It is plotted below.