

Lecture 17: Implicit differentiation

Nathan Pflueger

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1 Introduction

Today we discuss a technique called implicit differentiation, which provides a quicker and easier way to compute many derivatives we could already do, and also can be used to evaluate some new derivatives. The main example we will see of new derivatives are the derivatives of the inverse trigonometric functions.

Implicit differentiation is not a new differentiation rule; instead, it is a technique that can be applied with the rules we've already learned. The idea is this: instead of trying to tackle the desired function explicitly, instead just find a simple equation that the function satisfies (called an *implicit equation*). If you differentiate both sides of this equation, then you can usually recover the derivative of the function you actually cared about with just a little algebra.

The reference for today is Stewart section §3.5 (for implicit differentiation generally) and §3.6 (for inverse trig functions specifically).

2 Review: Inverse functions

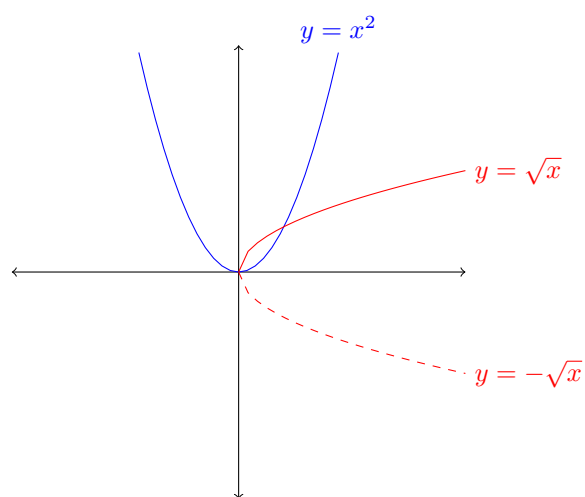
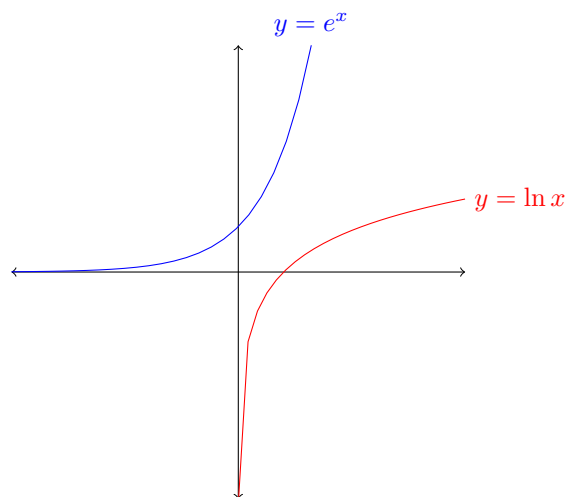
Functions like \sqrt{x} , $\ln x$, $\tan^{-1} x$ (which is also written $\arctan(x)$) and so forth are examples of what are called *inverse functions*. The idea is that they invert to the effect for some other function. In these cases:

- $(\sqrt{x})^2 = x$, so \sqrt{x} is the inverse of x^2 .
- $e^{\ln x} = x$, so $\ln x$ is the inverse of e^x .
- $\tan(\tan^{-1} x) = x$, so $\tan^{-1} x$ is the inverse of $\tan x$.

The general definition is: $g(x)$ is an inverse function of $f(x)$ if $f(g(x)) = x$ for all x in the domain of g .

Note. Many functions have more than one inverse function. For example, $-\sqrt{x}$ is also an inverse function of x^2 .

Graphically, you obtain inverse functions by reflecting the graph of the original function across the line $y = x$; that is, switch the roles of x and y , as in the following pictures.



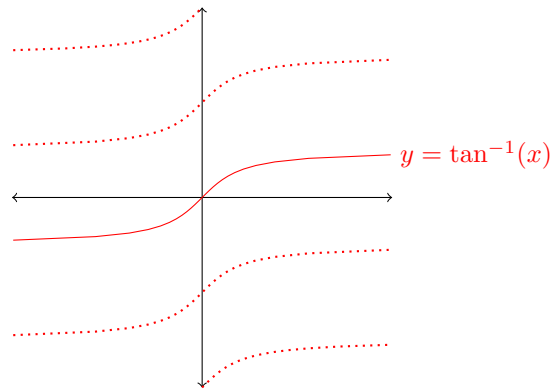
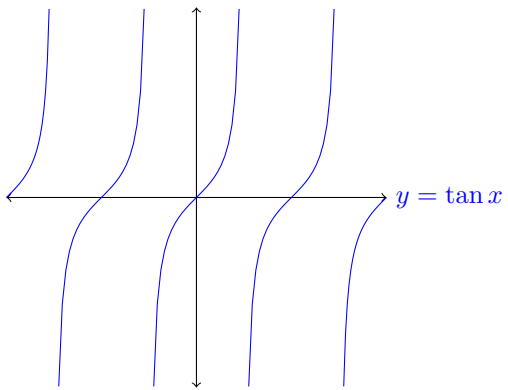
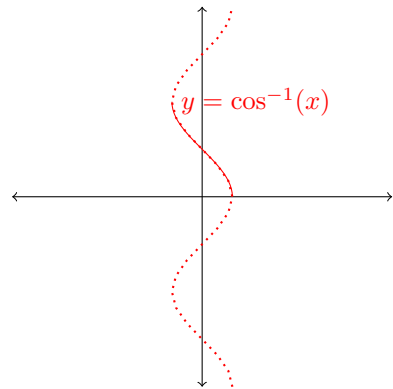
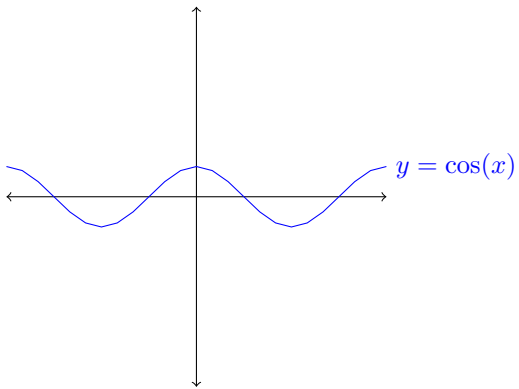
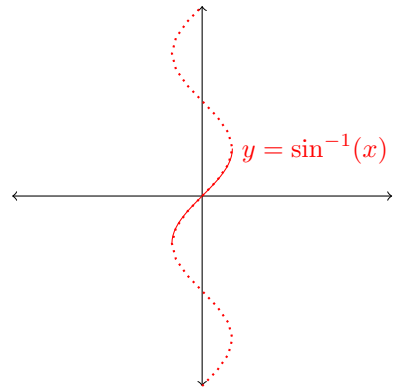
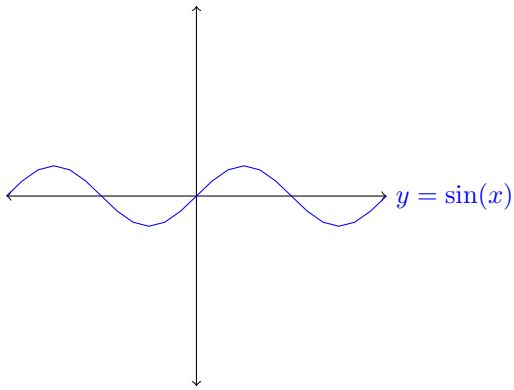
Note in the second picture that the two possible inverse functions *together* form the reflection of the original graph, but neither does individually.

The main inverse functions we are interested in are the inverse trigonometric functions. These are labeled on calculators by \sin^{-1} , \cos^{-1} , \tan^{-1} , and they are often called in other places by the names arcsin, arccos, arctan (there are also, of course, inverse functions of sec, csc, and cot, but we won't discuss these as much). In these cases, flipping the graph of the original functions give plots that have *many* y values of each x value, so there are many possible inverse functions. By convention, we simply choose one arc of each of these graphs to get the functions that we call \sin^{-1} , \cos^{-1} , \tan^{-1} . The following pictures show the arc chosen in solid red, and the rest of the "flipped graph" in dashed red.

Note in particular that the conventional "inverse trigonometric functions" have the following ranges.

It is confusing that the inverse cosine has a different range from the other two, but you can always visualize these pictures to remember which range should be which.

Function	Range
$\sin^{-1}(x)$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\pi/2, \pi/2)$



3 Derivatives of inverse functions

If you can differentiate a function, you can always write an expression for the derivative of its inverse. The technique is the first example of what we'll later call *implicit differentiation*. I'll illustrate the idea first by finding the derivative of the natural log function.

First, write the following equation.

$$e^{\ln x} = x$$

This equation can be regarded as basically just the *definition* of the natural logarithm. Except it doesn't define it *explicitly*; it defined it *implicitly*, but telling an equation it must satisfy. Now the technique is: differentiate both sides of this equation. Since the two sides are equal, their derivatives must be equal also.

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$

Now consider the two sides of this equation separately.

Left side: $\frac{d}{dx} e^{\ln x}$. This is a composition of two functions: $\ln x$ and e^x . So you can differentiate it with the chain rule, using the fact that $\frac{d}{dx} e^x = e^x$. In Leibniz notation, it looks like this.

$$\begin{aligned} \frac{d}{dx} e^{\ln x} &= \frac{d}{d(\ln x)} e^{\ln x} \cdot \frac{d}{dx} \ln x \quad (\text{chain rule}) \\ &= e^{\ln x} \cdot \frac{d}{dx} \ln x \quad (\text{derivative of } e^x) \\ &= x \cdot \frac{d}{dx} \ln x \quad (\text{since } e^{\ln x} = x) \end{aligned}$$

Right side: $\frac{d}{dx} x$. This is easy: the derivative of x is just 1.

$$\frac{d}{dx} x = 1$$

Putting these together: We can now just set these two expressions equal to each other, and then *solve the equation*.

$$\begin{aligned} \frac{d}{dx} e^{\ln x} &= \frac{d}{dx} x \quad (\text{starting point}) \\ x \frac{d}{dx} \ln x &= 1 \quad (\text{analysis of the two sides, above}) \\ \frac{d}{dx} \ln x &= \frac{1}{x} \quad (\text{divide both sides by } x) \end{aligned}$$

The result of this work is that, as if by magic, we were able to *solve* for the derivative of $\ln x$.

Aside. Here's a quick and easy way to remember the derivative of natural log, if you have a good visual imagination. If this doesn't make any sense to you, don't worry about; it isn't essential. The derivative of e^x is just e^x . That means that if (a, b) is any point on the graph $y = e^x$, then the slope of the tangent line is b . Now, "flip" this picture by switching x and y . That means that (b, a) is a point on the graph of natural log. The slope of the tangent line flips becomes $\frac{1}{b}$ (because rise and run have traded places). So: the slope of the tangent line to the graph $y = \ln x$ is just the reciprocal of the x -coordinate. This is the same thing as saying that $\frac{d}{dx} \ln x = \frac{1}{x}$. It is a good exercise to think this through and understand why this is logically equivalent to the algebra with the chain rule I've shown above.

3.1 Differentiating inverse trig functions

In this subsection, I'll show how to compute the derivatives of $\sin^{-1} x$ and $\tan^{-1} x$, using the same technique as I've shown above. As before, the method is: write down an equation satisfied by the function, differentiate both sides of this equation, and solve.

First, consider $\tan^{-1} x$. It is an inverse function of tangent; this just means that it satisfies this equation.

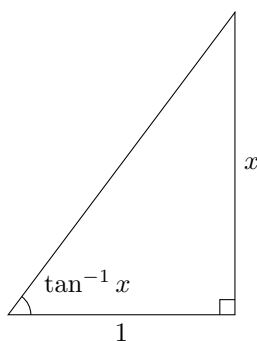
$$\tan(\tan^{-1} x) = x$$

Now, differentiate both sides of the equation. The right side is easy. The left side requires the chain rule. It also requires the fact that the derivative of $\tan x$ is $\sec^2 x$, which we found in a previous lecture using the quotient rule.

$$\begin{aligned}\frac{d}{dx} \tan(\tan^{-1} x) &= \frac{d}{dx} x \\ \sec^2(\tan^{-1} x) \frac{d}{dx} \tan^{-1} x &= 1 \quad (\text{Chain rule on the left; easy derivative on the right}) \\ \frac{d}{dx} \tan^{-1} x &= \frac{1}{\sec^2(\tan^{-1} x)}\end{aligned}$$

Now, we at least have a formula for the derivative of $\tan^{-1} x$, and in principle this means our work is done. But it is a good idea to simplify this expression slightly, because it will become something much simpler. To do this, remember exactly what $\tan^{-1} x$ is.

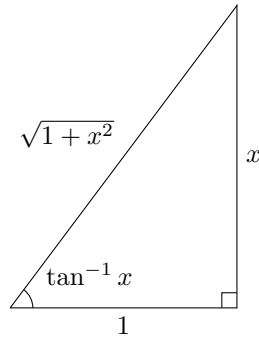
$\tan^{-1} x$ is precisely that angle in $(-\pi/2, \pi/2)$ whose tangent is x . So if we draw the following right triangle, the marked angle will have measure $\tan^{-1} x$ radians.



Remember: $\tan^{-1} x$ is an *angle*. Since $\tan x$ takes angles to ratios, \tan^{-1} takes ratios back to angles. In the triangle above, it is the angle marked with the little arc in the corner.

Where did this triangle come from? This was the most pressing question when we talked about this picture in class. The answer is: I made it up. It is a prop. But I've made it up with a specific purpose: it's purpose is to have the angle $\tan^{-1} x$ in it. The whole purpose of the triangle is to exhibit this angle so that I can study it.

Back to the main discussion, remember what we're trying to find here. We want to know what $\sec^2(\tan^{-1} x)$ is. I've drawn a triangle with the angle $\tan^{-1} x$ in it. Since the secant of an angle is $\frac{\text{hypotenuse}}{\text{adjacent}}$, we just need to find the hypotenuse of this triangle. We can get this from the Pythagorean theorem.



So the secant of the angle $\tan^{-1} x$ is precisely $\sqrt{1+x^2}/1 = \sqrt{1+x^2}$. So $\sec^2(\tan^{-1} x) = 1+x^2$. Using this, we can put the derivative of \tan^{-1} in its simplest form.

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}}$$

Alternative method. Recall that there is an identity (one of the many equivalent forms of the Pythagorean theorem) $\sec^2 x = \tan^2 x + 1$. We could also have used this identity to do this algebra, as follows:

$$\begin{aligned} \sec^2(\tan^{-1} x) &= 1 + \tan^2(\tan^{-1} x) \\ &= 1 + (\tan(\tan^{-1} x))^2 \\ &= 1 + x^2 \end{aligned}$$

Aside. Notice that *the only fact we've used about \tan^{-1} is that it is an inverse function of $\tan x$* . So actually, $\frac{1}{1+x^2}$ is the derivative of *any* inverse function of $\tan x$, not just the “standard” one that we call \tan^{-1} . If you look at the picture in the last section, you'll see why this makes sense: if you flip the graph of $y = \tan x$, there are many arcs, of which $y = \tan^{-1} x$ is only one, but all of them are just vertical translates of each other. So they all have the same derivative function.

Now let's move on to $\sin^{-1} x$. The technique is the same. Write an equation describing $\sin^{-1} x$ implicitly, and differentiate both sides of it. The right side is easy, and the left side uses the chain rule.

$$\begin{aligned} \sin(\sin^{-1} x) &= x \quad (\text{because it's an inverse function}) \\ \frac{d}{dx} \sin(\sin^{-1} x) &= \frac{d}{dx} x \quad (\text{differentiate both sides}) \\ \cos(\sin^{-1} x) \frac{d}{dx} \sin^{-1} x &= 1 \quad (\text{chain rule on left; easy derivative on right}) \\ \frac{d}{dx} \sin^{-1} x &= \frac{1}{\cos(\sin^{-1} x)} \quad (\text{divide on both sides}) \end{aligned}$$

Like before, in principle we're done here: we have a totally well-defined expression for the derivative of \sin^{-1} , which you can plug into a computer and everything. But again, it's worth re-expressing it in a simpler way. Like before, there are two standard ways to do this, which both just boil down to the Pythagorean theorem. The quickest one is to use the identity $\cos^2 x + \sin^2 x = 1$. Letting x be the angle $\sin^{-1} x$, this says:

$$\begin{aligned}
\cos^2(\sin^{-1} x) + \sin^2(\sin^{-1} x) &= 1 \\
[\cos(\sin^{-1} x)]^2 + x^2 &= 1 \\
[\cos(\sin^{-1} x)]^2 &= 1 - x^2 \\
\cos(\sin^{-1} x) &= \sqrt{1 - x^2}
\end{aligned}$$

So this gives the simplification that we want.

$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}}$$

Technical point I've swept under the rug: I took the square root of both sides of the equation $[\cos(\sin^{-1} x)]^2 = 1 - x^2$ above, to get $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$. But technically, all I could conclude here is $\cos(\sin^{-1} x) = \pm\sqrt{1 - x^2}$ (two numbers whose squares are equal are not necessarily equal: they are either equal or opposite). So how do we know that the plus sign is correct? Here is where we must use the specific definition of $\sin^{-1} x$: it doesn't return just any angle whose sine is x : it return the angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x . And the cosine of any angle in $[-\pi/2, \pi/2]$ is not negative. If we chose a different inverse function of $\sin x$, the derivative could be $-\frac{1}{\sqrt{1-x^2}}$ instead. Look at the picture in the previous section to see that this makes sense. If this confuses you, don't worry too much about it; I mention it for completeness.

Differentiating $\cos^{-1} x$ with this method is completely analogous. It is left to you as a homework problem.

3.2 Differentiating any inverse function

The method shown above works in general to differentiate any inverse function you like. Here's how it works: suppose that $f(x)$ is any function, and $f^{-1}(x)$ is an inverse function. Follow the same steps as above.

$$\begin{aligned}
f(f^{-1}(x)) &= x \text{ (inverse function)} \\
\frac{d}{dx} f(f^{-1}(x)) &= \frac{d}{dx} x \text{ (differentiate both sides)} \\
f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) &= 1 \text{ (chain rule)} \\
\frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))}
\end{aligned}$$

This last line is a valid formula for inverse function. Here are some examples.

Example 3.1. Suppose $f(x) = e^x$. Then $f'(x) = e^x$ and $f^{-1}(x) = \ln x$. So $\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$.

Example 3.2. Suppose $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1}(x)$. Then $\frac{d}{dx} \sin^{-1} x = \frac{1}{\cos(\sin^{-1}(x))}$, as in the last section.

Example 3.3. Suppose $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$. Then $f'(x) = 2x$, so $\frac{d}{dx} \sqrt{x} = \frac{1}{2f^{-1}(x)} = \frac{1}{2\sqrt{x}}$.

4 Implicit and explicit functions

In the previous sections, we've seen that a useful way to differentiate inverse functions like $\ln x$ or $\sin^{-1} x$ is not to attack them directly, but instead write some equation that describes them *implicitly*, differentiate both sides of the equation, and solve. We're now going to discuss this technique in more generality.

As a first step, I want to elaborate on what it means to write an implicit equation. An implicit equation is just an equation in two variables x and y that is not necessarily of the form $y = f(x)$. Here are some

examples.

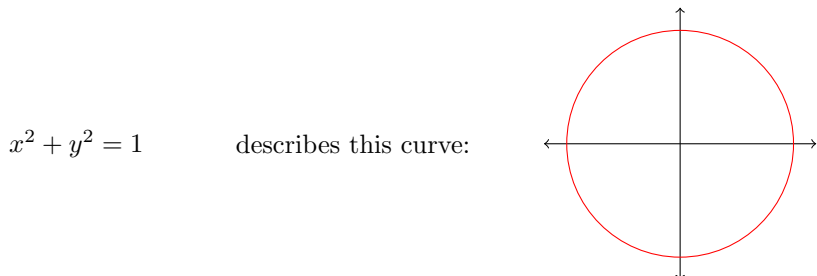
Explicit equations:

- $y = \sqrt{x}$
- $y = \sqrt{1 - x^2}$
- $y = \frac{x}{x-1}$

Implicit equations:

- $y^2 = x$
- $x^2 + y^2 = 1$
- $xy = x + y$

One major distinction between explicit and implicit equations is that implicit equations *don't necessarily describe graphs of functions*. Instead, they describe graphs of curves. Perhaps the simplest example is this implicit equation, which defines a circle.



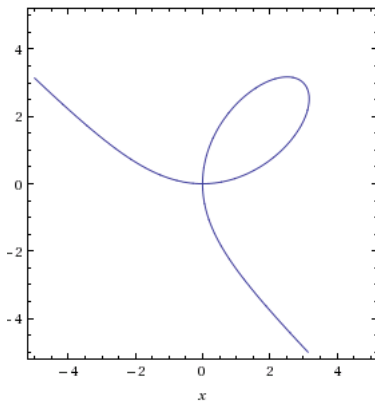
Now there are two graphs that lie on the circle above: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ (one is the upper semicircle, and one is the lower semicircle). So the *implicit* equation $x^2 + y^2 = 1$ describes two different *explicit* equations. This is often the case with implicit equations.

Here are two more examples of implicit equations, which we'll revisit in the last section.

Example 4.1. (Descartes' leaf) Consider the following equation.

$$x^3 + y^3 = 6xy$$

This is an implicit equation. It describes the curve shown below (made with Wolfram alpha).



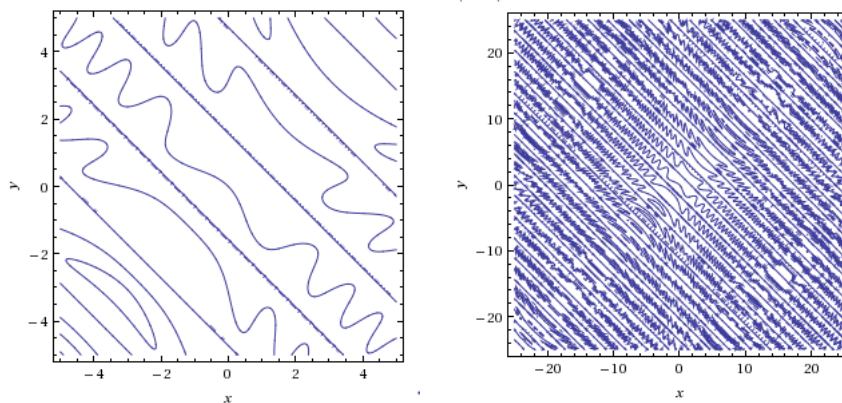
This curve is traditionally called the “folium of Descartes” (“folium” is Latin for “leaf”). This equation first occurred in a letter from Descartes to Fermat (Fermat was a lawyer by profession, but did mathematics as a hobby). Fermat claimed to have a method to find tangent lines to any curve, and Descartes invented this curve as a challenge to Fermat. Fermat was successful in finding tangent lines to the curve. Today, however, the problem of finding tangent lines is very easy, due to the invention of calculus some time later. This event was notable enough in the history of calculus that it is memorialized on the following Albanian postage stamp¹.



Example 4.2. Consider the following implicit equation.

$$\tan(x + y) = \sin(xy)$$

If you graph its solution curve, it is the following very complex picture (shown at two different levels of zoom).



As you can see, there are *many* different functions that obey this implicit equation, because there are many different values of y for a given value of x .

¹I am not aware of any relation between Descartes and Albania, but I only spent a couple minutes googling for it.

5 Implicit differentiation

Now we'll look at how you can use an implicit equation to find derivatives. The basic technique will always be as follows.

- Differentiate both sides of the implicit equation. Remember that y is a function of x .
- Use algebra to solve for $\frac{dy}{dx}$. The result will be an expression in x and y .

Let's begin with a simple example.

Example 5.1. Consider the implicit equation of a circle.

$$x^2 + y^2 = 1$$

Now imagine that y is some function of x that obeys this implicit equation. We can differentiate both sides of the equation with respect to x .

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}1 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y \cdot \frac{dy}{dx} &= 0 \quad (\text{chain rule}) \\ 2y \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{2y}{2x} \\ \frac{dy}{dx} &= -\frac{y}{x}\end{aligned}$$

This shows that the slope of the tangent line to the circle described by $x^2 + y^2 = 1$ is always given by $-y/x$ at a point (x, y) .

Note. When you solve for $\frac{dy}{dx}$, it will almost always be an expression in terms of both x and y . In some cases, you can re-express it as something purely in terms of x , but not always. The next example shows one case where you can re-express it.

Example 5.2. Suppose that $y = \sqrt[3]{\cos x + 7}$. Find $\frac{dy}{dx}$ using implicit differentiation.

Note. You can differentiate this using the chain rule as well. Of course you will get the same answer as we get below, but you may find one technique or the other easier.

Solution. Cube both sides to obtain $y^3 = \cos x + 7$. Differentiate both sides:

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(\cos x + 7) \quad (\text{differentiate both sides}) \\ 3y^2 \frac{dy}{dx} &= -\sin x \quad (\text{chain rule on the left, known derivatives on the right}) \\ \frac{dy}{dx} &= -\frac{\sin x}{3y^2} \quad (\text{divide both sides by } 3y^2)\end{aligned}$$

In this case, we have an explicitly equation for y , namely $y = \sqrt[3]{\cos x + 7}$, so we can substitute that back into the answer here to get the derivative purely in terms of x .

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\sin x}{3(\sqrt[3]{\cos x + 7})^2} \\ &= -\frac{\sin x}{3(\cos x + 7)^{3/2}}\end{aligned}$$

To summarize, there are two main reasons to differentiate implicit equations.

1. Because you have no explicit equation for y in terms of x (you can still obtain $\frac{dy}{dx}$ in terms of x and y).
2. Because you have an explicit equation, but an implicit equation is much simpler (in this case, you can substitute your explicit equation back at the end to get $\frac{dy}{dx}$ purely in terms of x).

6 Examples

As examples of the technique of implicit differentiation, we will find some tangent lines to the two implicit curves given in section 4.

Example 6.1. Consider the equation for Descartes' leaf.

$$x^3 + y^3 = 6xy$$

1. Find the tangent line to this curve at the point $(3, 3)$.
2. Find the tangent line to this curve at the point $(\frac{4}{3}, \frac{8}{3})$.

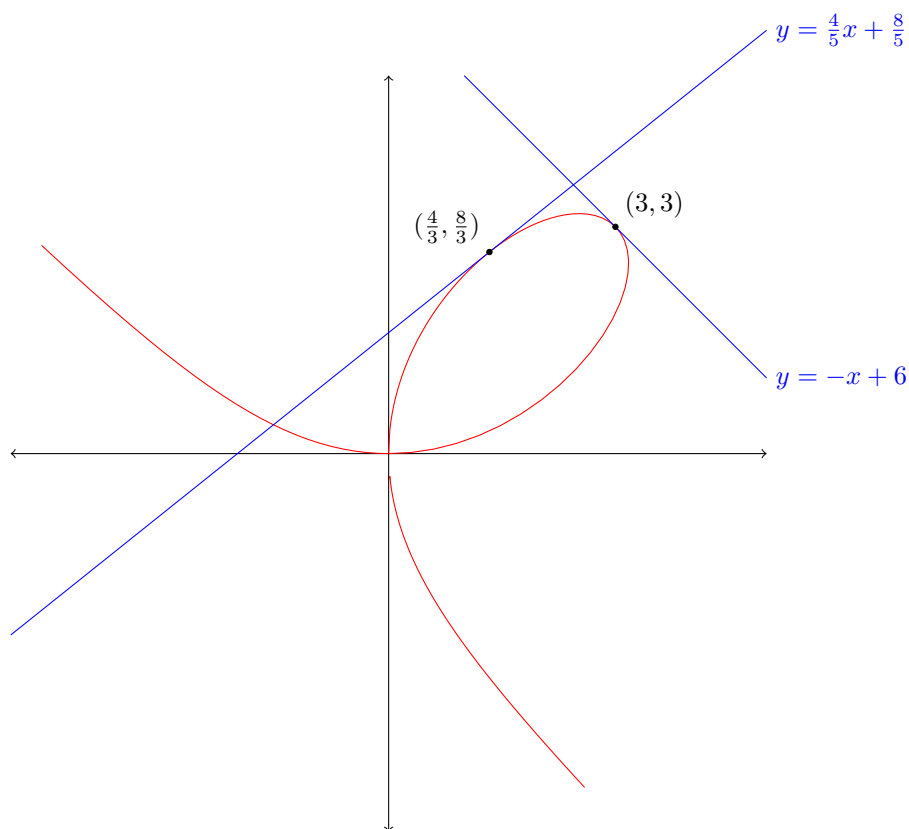
Solution. Begin by differentiating both sides of the equation, as functions of x .

$$\begin{aligned}\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6xy) \\ 3x^2 + 3y^2 \frac{dy}{dx} &= 6 \frac{dx}{dx} y + 6x \frac{dy}{dx} \quad (\text{product rule used on the right side}) \\ 3x^2 + 3y^2 \frac{dy}{dx} &= 6y + 6x \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} &= 6y - 3x^2 \quad (\text{move all the } \frac{dy}{dx} \text{ to one side}) \\ (3y^2 - 6x) \frac{dy}{dx} &= 6y - 3x^2 \quad (\text{group like terms}) \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \quad (\text{divide on both sides}) \\ \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x} \quad (\text{cancel the factor of 3 on top and bottom})\end{aligned}$$

Now, we can use this expression to compute the two desired tangent lines.

1. At the point $(3, 3)$, the slope of the tangent line is $\frac{dy}{dx} = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = \frac{-3}{3} = -1$. So the tangent line is the line with slope -1 through the point $(3, 3)$. Therefore this line is given by the equation $\boxed{(y - 3) = (-1)(x - 3)}$, or alternatively $\boxed{y = -x + 6}$.
2. At the point $(\frac{4}{3}, \frac{8}{3})$, the slope of the tangent line is given by $\frac{2 \cdot \frac{8}{3} - (\frac{4}{3})^2}{(\frac{8}{3})^2 - 2 \cdot \frac{4}{3}}$. This simplifies to $\frac{16/3 - 16/9}{64/9 - 8/3} = \frac{48/9 - 16/9}{64/9 - 24/9} = \frac{32/9}{40/9} = \frac{32}{40} = \frac{4}{5}$. So the equation of the tangent line is $\boxed{(y - \frac{8}{3}) = \frac{4}{5}(x - \frac{4}{3})}$, or if you prefer, $\boxed{y = \frac{4}{5}x + \frac{8}{5}}$.

The curve, with these two tangent lines, is shown below.



Example 6.2. Consider the curve given by the following implicit equation.

$$\tan(x + y) = \sin(xy)$$

Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

Solution. Begin by differentiating both sides of the equation. As usual, regard y as a function of x .

$$\begin{aligned} \tan(x + y) &= \sin(xy) \\ \frac{d}{dx} \tan(x + y) &= \frac{d}{dx} \sin(xy) \\ \sec^2(x + y) \frac{d}{dx} (x + y) &= \cos(xy) \frac{d}{dx} (xy) \quad (\text{chain rule on both sides}) \\ \sec^2(x + y) \left(\frac{dx}{dx} + \frac{dy}{dx} \right) &= \cos(xy) \left(\frac{dx}{dx} y + x \frac{dy}{dx} \right) \quad (\text{product rule on the right}) \\ \sec^2(x + y) \left(1 + \frac{dy}{dx} \right) &= \cos(xy) \left(y + x \frac{dy}{dx} \right) \quad (\text{because } \frac{dx}{dx} = 1) \end{aligned}$$

So far so good. Now solve this whole monster for $\frac{dy}{dx}$.

$$\begin{aligned} \sec^2(x+y) + \sec^2(x+y) \frac{dy}{dx} &= y \cos(xy) + x \cos(xy) \frac{dy}{dx} \quad (\text{distributing terms}) \\ (\sec^2(x+y) - x \cos(xy)) \frac{dy}{dx} &= y \cos(xy) - \sec^2(x+y) \quad (\text{moving all the } \frac{dy}{dx} \text{ to one side}) \\ \frac{dy}{dx} &= \frac{y \cos(xy) - \sec^2(x+y)}{\sec^2(x+y) - x \cos(xy)} \quad (\text{divide on both sides}) \end{aligned}$$

To find the slope at the specific point $(\sqrt{\pi}, -\sqrt{\pi})$, just substitute the values of x and y into this expression. Notice first of all that $xy = -\pi$ and $x + y = 0$; these occur in multiple places in the expression.

$$\begin{aligned} \text{slope} &= \frac{-\sqrt{\pi} \cos(-\pi) - \sec^2(0)}{\sec^2(0) - \sqrt{\pi} \cos(-\pi)} \\ &= \frac{-\sqrt{\pi} \cdot (-1) - 1}{1 - \sqrt{\pi} \cdot (-1)} \\ &= \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1} \\ &\approx 0.28 \end{aligned}$$

Therefore the equation of the tangent line is $(y + \sqrt{\pi}) = \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1} \cdot (x - \sqrt{\pi})$. In slope-intercept form,

this is $y = \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1}x - \frac{2\pi}{\sqrt{\pi} + 1}$. If you use a calculator to approximate all these numbers, you can also write this equation as $(y + 1.77) = 0.28 \cdot (x - 1.77)$ or $y = 0.28x - 2.27$ (any of these four boxed answers would be acceptable on homework). This line is shown on the plot below (Wolfram alpha has marked all the points of intersection; the point $(\sqrt{\pi}, -\sqrt{\pi})$ is the second from the right, where the line is tangent to the curve).

