# Lecture 22: Related rates

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## 1 Introduction

Today we consider some problems in which several quantities are changing over time. These problems are called "related rates" problems, because the rates of change of the various quantities will be related in some specific way. These problems are very similar to implicit differentiation problems: the basic method is to write an equation of functions, and differentiate both sides, then solve the result for the desired quantity.

The reference for today is Stewart §4.1.

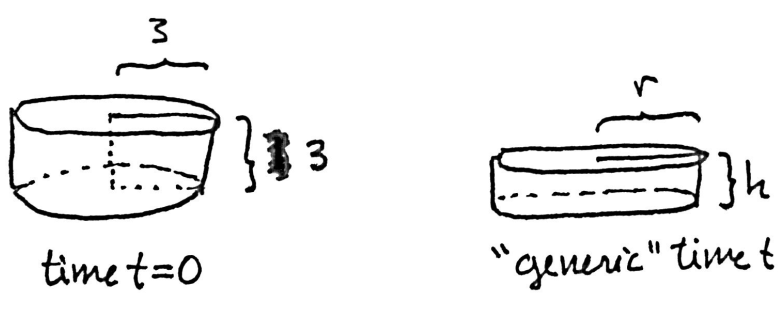
## 2 First example: a melting cylinder

I will illustrate the basic steps in related rates with the following problem.

*Problem.* Suppose that a disc of very hot metal is slowly flattening out. As it flattens, it is always shaped like a cylinder. Suppose that at some instant, the cylinder is $3cm$ in height, with a radius of $3cm$. Suppose also that the height of the cylinder is decreasing at a rate of 0.1 centimeters per second. How quickly is the radius increasing?

First I'll give a rather detailed and verbose solution to show all the steps. Then I will rewrite this solution in a shorter way, along the lines of what you can write in your homework.

*Solution (in detail).* Begin by **identifying the quantities involved.** There are two choices that are clearly important: the height and radius of the cylinder. Both of these are changing with time, we let's write them as functions: $h(t)$ and $r(t)$ (in later problems, and also in the brief solution, we'll tend to just write $h$ and $r$ without the $(t)$; but for now I'll leave it in to emphasize that these are functions of time). We know that at the instant in question, the height is decreasing by 0.1 centimeters per second. We can display these quantities in the following picture.




Next we need to **relate these quantities.** In this case, they are constrained by the fact that the volume of the cylinder doesn't change. We know that the volume of a cylinder is always $\pi r^2 h$. In fact, we know that at the instant in question, the radius and height are both 3, so this volume must be $\pi 3^2 \cdot 3 = 27\pi$. This must equal the volume at all times, so we can write the following equation.

$$\pi r(t)^2 h(t) = 27\pi$$

Now, what information do you know, exactly? We know that at some instant, $r$ is 3 and $h$ is 3. Let's take this instant to be $t = 0$: so $r(0) = 3$ and $h(0) = 3$. We also know that the height is decreasing by 0.1 centimeters per second; in symbols, $h'(0) = -0.1$. We want to somehow obtain $r'(0)$ from this information (the rate that the radius is increasing). To do this, we **differentiate both sides** of our constraining equation. This involves the product rule, as follows.

$$\begin{aligned} \frac{d}{dt}\left(\pi r(t)^2 h(t)\right) &= \frac{d}{dt} 27\pi \\ \pi \cdot 2r(t)r'(t)h(t) + \pi r(t)^2 h'(t) &= 0 \quad \text{(product rule and derivative of a constant)} \end{aligned}$$

From here, we can just plug in $t = 0$ and use the facts that we already know, namely:

- $r(0) = 3$
- $h(0) = 3$
- $h'(0) = -0.1$

And from this we obtain:

$$\begin{aligned} \pi \cdot 2r(0)r'(0)h(0) + \pi r(0)^2 h'(0) &= 0 \\ \pi \cdot 6r'(0) \cdot 3 + \pi \cdot 3^2 \cdot (-0.1) &= 0 \\ 18\pi r'(0) - 0.9\pi &= 0 \\ 18\pi r'(0) &= 0.9\pi \\ r'(0) &= \frac{0.9\pi}{18\pi} \\ &= 0.05 \end{aligned}$$

Therefore we were able to **solve for the desired rate** by doing some algebra. The result is that the radius of the cylinder is increasing at a rate of 0.05 centimeters per second.

*Solution (more brief version)* Let $r, h$, and $V$ be the radius, height, and volume of the cylinder at a given time. We know that at all times, the volume $\pi r^2 h$ is constant; call this constant value $V$. Then we can write the following constraint and differentiate it.

$$\begin{aligned} \pi r^2 h &= V \\ \pi (r^2 h)' &= 0 \quad \text{(since } V \text{ is constant)} \\ \pi (2r \cdot r' \cdot h + r^2 h') &= 0 \\ 2r \cdot r' \cdot h + r^2 h' &= 0 \\ 2rh \cdot r' &= -r^2 h' \\ r' &= -\frac{r^2 h'}{2rh} \end{aligned}$$

At the instant in question, we have $r = 3, h' = - - 0.1$, and $h = 3$. Therefore at this moment, $r' = -\frac{3^2 \cdot (-0.1)}{2 \cdot 3 \cdot 3} = \frac{0.9}{18} = 0.05$. So the radius is expanding by 0.05 centimeters per second.

**Note.** One difference between these two solutions is that in the second solution I completely solved for the desired rate $r'$ before substituting any numbers. It is up to you whether to solve or substitute numbers first. Sometimes one method will take a little less work than the other, but it depends.

# 3 The general strategy

Roughly speaking, here are the necessary steps to solve a related rates problem.

1. Draw two pictures: one of the situation at the instant you're studying, and one showing the "generic" situation.
2. Identify the quantities involved, give them names.
3. Write an equation relating the quantities in question.
4. Differentiate both sides of the equation, to get an equation relating the quantities and their rates of change.
5. Solve for the rate of change you are interested in.

Like many other types of problems we study in this class, you will sometimes want to modify this strategy to suit the particular needs of the problem.

# 4 More examples

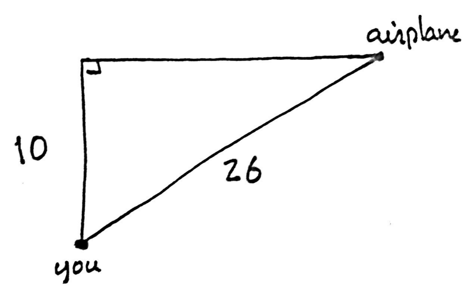
You can find many more examples in the textbook and on the various worksheets on the website.

*Example* 4.1. Suppose that you are standing on the ground, under the flight path of an airplane. The airplane is flying at an altitude of 10 km. Using a radar device, you are able to detect that the plane is currently 26 km away from you, and that its *distance to you* is currently shrinking at a rate of 840 kilometers per hour.

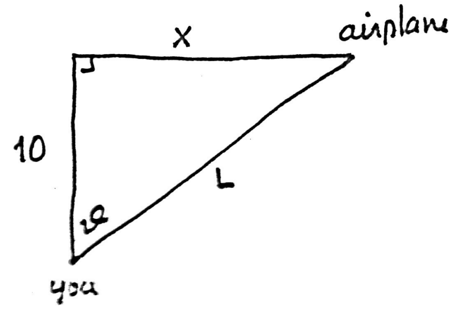
(a) What is the current speed of the airplane?

(b) Let $\theta$ be the angle your arm makes with the ground if you point the radar device straight at the plane. How quickly is $\theta$ increasing?

*Solution.*
Here's a rough sketch of the current situation. Distances are measured in kilometers.



Let's define some notation: let $L$ be the distance from you to the airplane, $x$ be the distance from the airplane to the spot right above your head (in its flight path), and $\theta$ be the angle shown.




(a) To find the airspeed of the airplane, we need to know the rate of change of the number $x$. Let's relate it to the other quantities, using the Pythagorean theorem, and then differentiate. Remember that both $x$ and $L$ are functions of time!

$$\begin{aligned} 10^2 + x^2 &= L^2 \\ (10^2 + x^2)' &= (L^2)' \\ 2x \cdot x' &= 2L \cdot L' \end{aligned}$$

Now, the rate we want is the derivative $x'$, since since measure how fast the plane is actually traveling through the air. So solve for this quantity.

$$\begin{aligned} x' &= \frac{2L \cdot L'}{2x} \\ &= \frac{L \cdot L'}{x} \end{aligned}$$

We know that at the instant in question, $L = 26$ and $L' = -840$. We can also compute $x$ at this time: $x = \sqrt{L^2 - 10^2} = \sqrt{26^2 - 10^2} = 24$. Therefore we can compute $x'$ as follows.

$$\begin{aligned} x' &= \frac{26 \cdot (-840)}{24} \\ &= -910 \end{aligned}$$

So $x$ is decreasing at a rate of 910 kilometers per hour. That means that the airplane is flying at $\boxed{\text{910 kilometers per hour}}$.

(b) Write the following equation to relate the angle $\theta$ to $L$, then differentiate and solve for $\theta'$.

$$\begin{aligned} 10 &= L\cos\theta \\ 0 &= L'\cos\theta + L(-\sin\theta)\theta' \\ L\sin\theta \cdot \theta' &= L'\cos\theta \\ \theta' &= \frac{L'\cos\theta}{L\sin\theta} \end{aligned}$$

We can also re-express the quantity $L\sin\theta$ as $x$ and the quantity $\cos\theta$ as $10/L$, so obtain the following.

$$\theta' = \frac{10L'}{Lx}$$

Now we can substitute known values: $L = 26, L' = 840$ and $x = 24$ to obtain $\theta' = \frac{10\cdot(-840)}{26\cdot 24} = -13.46$ (radians per hour).

Finally, note that the angle made by your arm while holding the instrument is merely $\frac{pi}{2} - \theta$, so this angle is increasing at a rate of $\boxed{\text{13.46 radians per hour}}$. These are not very easy units to visualize; if you are curious, this is the same thing as about 13 degrees per minute, or 0.2 degree per second. So when the plane still 24 kilometers out, you must only raise your arm 1 degree every 5 seconds to track it, roughly. Note that the plane will eventually pass over your head after about 90 seconds, which shows that this angle will certainly start changing much more quickly fairly soon.

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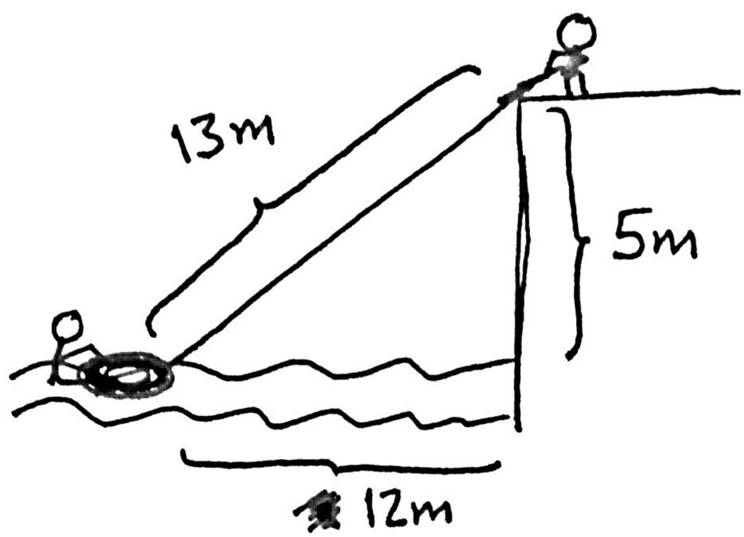
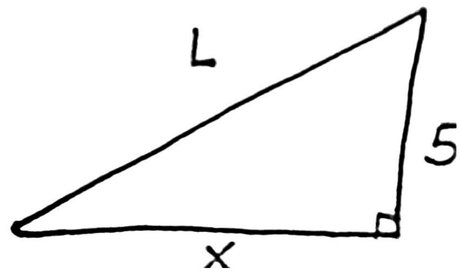
*Example* 4.2. Suppose you are standing on the deck of a boat, 5 meters above the surface of the water. In the water is a person who has fallen overboard, whom you are pulling back to the boat with a rope. Suppose that the person is currently 12 meters from the side of the boat, and they are moving at 1 meter per second through the water to the side of the boat. How quickly are you pulling in the rope, in meters per second?

*Solution.* Here is a rough sketch of the situation at this moment. The current length of the rope can be computed from the Pythagorean theorem to be 13 meters.




Here is a picture of the relevant quantities in this problem, at any given time.




So at the instant in question, $L = 13$ and $x = 12$. We also know that the person is moving towards the boat at 1 meter per second, i.e. $x' = -1$ at this instant. We want to determine $L'$, the rate that the length of rope is changing. To do this, relate these quantities using the Pythagorean theorem, differentiate both sides, and solve.

$$\begin{aligned} x^2 + 5^2 &= L^2 \\ 2x \cdot x' + 0 &= 2L \cdot L' \\ \frac{x \cdot x'}{L} &= L' \end{aligned}$$

Substituting the known values at this moment, we see that the rate of change of the rope's length is $L' = \frac{12 \cdot (-1)}{13} = -\frac{12}{13} \approx 0.92$ meters per second. So you are pulling in about 0.92 meters of rope per second.

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*Example* 4.3. Suppose that a balloon filled with gas is slowly expanding. Currently, the pressure inside the balloon is $100,000$ Pa, the volume of gas is 1 cubic meter, and the temperate of the gas is 27 degrees Celsius. Suppose also that the volume is increasing at a rate of 0.1 cubic meters per second, and the temperature is increasing by 3 degrees Celsius per second. Is the pressure in the balloon increasing or decreasing? How quickly?
*Hint.* If $P, V, T$ are the pressure, volume, and temperature (in Kelvin) of a gas, then under reasonable conditions these numbers obey the *ideal gas law* $PV = nRT$, where $n$ measures the number of gas molecules and $R$ is a constant.

*Solution.* One approach is to differentiate the ideal gas law. This will work, of course, but it turns out that there's a much cleaner way: first the the logarithm, and then differentiate. You could guess that this will be easier this both sides of the gas law come from multiplying several things.

Taking the logarithm and differentiating, using the fact that $nR$ is constant, gives the following.

$$\begin{aligned} \ln(PV) &= \ln(nRT) \\ \ln P + \ln V &= \ln(n) + \ln(R) + \ln T \quad \text{(rules of logarithms)} \\ \frac{P'}{P} + \frac{V'}{V} &= \frac{T'}{T} \quad \text{(differentiate both sides; use the fact that } n, R \text{ are constant)} \end{aligned}$$

Let's summarize the known information, at the moment we are studying. Note in particular that since $T$ should be expressed in Kelvin (not degrees Celsius), is is $273 + 27 = 300$. Since a change of 3 degrees Celsius is the same as a change in 3 degrees Kelvin, however, $T' = 3$ in either unit system.

- $P = 10^5$
- $V = 1$

- $V' = 0.1$
- $T = 300$
- $T' = 3$

We can plug all of this information in, and easily solve for the desired rate of change $P'$ as follows.

$$\begin{aligned}
\frac{P'}{P} + \frac{V'}{V} &= \frac{T'}{T} \\
\frac{P'}{10^5} + \frac{0.1}{1} &= \frac{3}{300} \\
\frac{P'}{10^5} &= 0.01 - 0.1 \\
P' &= -10^5 \cdot 0.09 \\
P' &= -9000
\end{aligned}$$

So the pressure in the balloon is currently $\boxed{\text{decreasing by 9000 Pascals per second}}$.