Lecture 25: Antiderivatives

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8 November 2013

1 Introduction

Today we begin the second main part of calculus: integration. This is a subject that is covered in much greater detail in Math 1B; we will cover the basic aspects of it in this course. We begin today with the notion of an antiderivative, which is roughly what it sounds like: given a function, we attempt to realize it as the derivative of some other function. That other function is called the antiderivative.

The topic of integration is enormous, due to the many techniques and patterns that have been found by various mathematicians and scientists over time. Today we will merely present it mainly as a guessing game, and introduce some of the basic vocabulary. Next week we will discuss the role that antiderivatives play in certain geometric problems.

The reference for today is Stewart §4.8.

2 Terminology

Definition 2.1. A function $F(x)$ is called an *antiderivative* of another function $f(x)$ if and only if $F'(x) = f(x)$.

There is a fairly common notional convention to use uppercase letters for antiderivatives, and lowercase letters for the original functions (so $G(x)$ would be an antiderivative of $g(x)$). This convention is great on paper, but can lead to confusion when speaking out loud. In any case, we will generally follow it in this course.

Here are some basic examples.

function	(an) antiderivative
$f(x) = x$	$F(x) = \frac{1}{2}x^2$
$f(x) = e^x$	$F(x) = e^x$
$f(x) = e^x$	$F(x) = e^x + 27$
$f(x) = 7e^x$	$F(x) = 7e^x$
$f(x) = 0$	$F(x) = 1$
$f(x) = 0$	$F(x) = 7$
$f(x) = 0$	$F(x) = 0$

An important observation here is that a given function always has more than one antiderivative. However, the antiderivatives are all related in a very simple way: they differ only by the addition of a constant.

This is the reason behind the following convention: when we talk about **the** antiderivative of a function, we will always write it in an expression like this:

The antiderivative of $f(x) = x$ is $F(x) = \frac{1}{2}x^2 + C$.

Here, the letter C is a constant, but it can be chosen to be any constant. The various different values of C give all possible antiderivatives of $f(x)$. There is an extra credit problem on your homework meant to help you understand why every antiderivative is guaranteed to arise this way (as a constant plus any other antiderivative).

To summarize:

- If we ask for "an antiderivative" of some function $f(x)$, we are asking for any function $F(x)$ such that $F'(x)$. It doesn't matter how you find it; all that matters is that its derivative should be $f(x)$.
- If we ask for "the antiderivative" of some function $f(x)$, we mean an expression of the form $F(x) + C$. So literally find any antiderivative you like, and just write "$+C$" at the end of it.

To be honest, I think calculus teachers are pretty overzealous about this distinction (and too picky about always writing "$+C$" at the end of antiderivatives). Also to be honest, I will definitely forget sometimes in class, and you should remind me. But the good news is that it isn't any extra work to write the general antiderivative once you've found even a single function: just write $+C$ after it and you are done.

The antiderivative is sometimes also called the *indefinite integral* of the function; we won't use this term much today but it will become more important later.

The final bit of notation I'll introduce today is the symbol $\int$. This symbol is used in the following way.

$$\int f(x)dx = F(x) + C$$

That is, the expression $\int f(x)dx$ means the same thing as "the antiderivative of $f(x)$" (complete with the $+C$ at the end). This symbol is meant to be a long, drawn out "S"; the S is for "sum," for reasons that will become clear next week. Also, the mysterious symbol dx here will become clear later; for now, just think of it as a little flag reminding you that x is the variable you are thinking about, just like in the symbol $\frac{d}{dx}$ it reminds you to differentiate with respect to x (rather than, for example, t).

3 Some common antiderivatives

Here is a summary of some of the most common antiderivatives you will encounter. Most of these are frequent enough that you should be able to recall them instantaneously (you should either memorize them, or practice deriving them enough that you don't need to think about it).

Polynomials. For these, remember that we have the power rule: $\frac{d}{dx}x^n = nx^{n-1}$. If you divide both sides by n you obtain $\frac{d}{dx}\frac{1}{n}x^n = x^{n-1}$. Just by running this rule backwards, you can obtain the following antiderivatives.

$$\begin{aligned}
\int 1dx &= x + C \\
\int xdx &= \frac{1}{2}x^2 + C \\
&\text{generally:} \\
\int x^n dx &= \frac{1}{n+1}x^{n+1} + C \quad \text{(for all } n \neq -1\text{)}
\end{aligned}$$

To anti differentiate general polynomials, just look at each term individually. For example:

$$\begin{aligned}
\int \left(7x^3 + 1 + x^{-2}\right) dx &= \frac{7}{4}x^4 + \frac{1}{1}x + \frac{1}{-1}x^{-1} + C \\
&= \frac{7}{4}x^4 + x - x^{-1} + C
\end{aligned}$$

1/x. Notice that the power rule tells you the antiderivative of x^n for all n except $n = -1$ (since you can't divide by 0). So what's the missing piece?

Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$. So $\ln x$ is an antiderivative of $\frac{1}{x}$. There's only one problem: $\ln x$ is only defined for positive numbers! What if I want an antiderivative that is defined for negative numbers, too? It turns out that by the chain rule, $\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$. In other words: $\ln(-x)$ (which is defined only for negative numbers) is an antiderivative for $\frac{1}{x}$ also. It is standard to combine these two facts, and write the following.

$$\int \frac{1}{x} dx = \ln |x| + C$$

This is confusing, unfortunately; it may be easiest for you to just memorize the equation above. We haven't seen these absolute value signs before since we haven't needed them. The only reason they are there now is so that we can have a nice antiderivative defined everywhere that $\frac{1}{x}$ is defined.

By the way, this is sometimes written $\int \frac{dx}{x} = \ln |x| + C$, by acting as if dx is a number. Next week we may discuss why this is common (it comes from the roots of the subject, when people really *did* think of dx like an "infinitely small number").

Exponential functions. Remember that $\frac{d}{dx} e^x = e^x$ and $\frac{d}{dx} a^x = \ln a \cdot a^x$ for any other constant a. From this it is easy to guess the antiderivatives of exponential functions: just divide by an appropriate constant.

$$\begin{aligned} \int e^x dx &= e^x + C \\ \int a^x dx &= \frac{1}{\ln a} a^x + C \end{aligned}$$

Trigonometric functions. The usual trigonometric derivatives give the following antiderivatives.

$$\begin{aligned} \int \cos x dx &= \sin x + C \\ \int \sin x dx &= -\cos x + C \\ \int \sec^2 x dx &= \tan x + C \\ \int \sec x \tan x dx &= \sec x + C \end{aligned}$$

You may wonder about a few conspicuous absences: what is $\int \tan x dx$, for example? It's a little harder to guess than the others, and we'll see a way to find it later.

Inverse trigonometric. Here are the main two inverse trig functions' derivatives, written backwards as antiderivatives.

$$\begin{aligned} \int \frac{dx}{1+x^2} &= \tan^{-1} x + C \\ \int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x + C \end{aligned}$$

Note the funny notation: I've taken the dx and put it in the numerator of the fractions $\frac{1}{1+x^2}$ and $\frac{1}{\sqrt{1-x^2}}$. This is common practice, even though the dx isn't a number. So take note of it.

A general warning: It is very common for students to mix up whether to multiply or divide by constants when doing antiderivatives (for example: is the antiderivative of x^7 equal to $8x^8$ or $\frac{1}{8}x^8$?). Here's a rule of thumb: when differentiating, you tend to multiply by constants (so $\frac{d}{dx}x^{10} = 10x^9$), but when antidifferentiating you tend to divide (so $\int x^{10}dx = \frac{1}{11}x^{11} + C$). But always remember, you can use differentiate rules to check anti differentiation rules; if you remember one you can check your memory of the other.

4 First example: falling on the moon.

Here's a sample problem, to illustrate a basic application of antiderivatives.

Problem. Suppose that you are standing on a ledge on the moon, 5 meters above the ground, and you jump off the edge. How long will it take for you to reach the ground? How fast will you hit?
Note: According to a quick google search, acceleration due to gravity on the moon is about $1.6m/s^2$.

I chose the moon for this problem since there is no air resistance, so that this analysis will actually be almost exactly correct (even if you were a feather, a beach ball, a cannon ball, or anything else).
Solution.

Begin by defining the following function.

$$h(t) = \text{distance from the ground in meters, } t \text{ seconds after jumping}$$

The method of solution will be: determine a formula for the function $h(t)$, from the given information. Then solve the equation $h(t) = 0$ for t to determined the time of impact. The speed of impact will come from evaluating $h'(t)$ at that time.
Now, we know the following physical interpretations for the first two derivatives of this function.

$$\begin{aligned} h'(t) &= \text{velocity after } t \text{ seconds (negative numbers mean downward motion)} \\ h''(t) &= \text{acceleration after } t \text{ seconds (negative numbers means towards the ground)} \end{aligned}$$

We know the following information about this function: when you are in free fall, gravity accelerated you towards the ground at a constant rate. That is:

$$h''(t) = -1.6$$

So the second derivative actually doesn't depend on t. Also, we know that when you first go over the ledge, you haven't started falling yet (you are like Wile E. Cayote, having just realized he is no longer over solid ground). Also, at that time you are 5 meters from the ground. Therefore

$$\begin{aligned} h'(0) &= 0 \\ h(0) &= 5 \end{aligned}$$

We also know that $h'(t)$ is an antiderivative for $h''(t)$, and $h(t)$ is an antiderivative for $h'(t)$.
Since $h'(t)$ is an antiderivative for the constant function -1.6, we know that $h'(t) = -1.6t + C$, for some constant C. Since $h'(0) = 0$, we know that $-1.6 \cdot 0 + C = 0$, i.e. $C = 0$ and $h'(t)$ is just $h'(t) = -1.6t$.

Since $h(t)$ is an antiderivative for $-1.6t$, it must be $-1.6 \cdot \frac{1}{2}t^2 + C_2 = -0.8t^2 + C_2$ for some constant C_2. Since $h(0) = 5$, we must have $-0.8 \cdot 0 + C_2 = 5$, i.e. $C_2 = 5$. So in fact $h(t) = -0.8t^2 + 5$.

So the time of impact is given by solving $h(t) = 0$, or $5 = 0.8t^2$, which gives $t = \pm\sqrt{6.25} = \pm 2.5$. Negative numbers are not physically relevant here, so $t = 2.5$ is the solution we want. Therefore $\boxed{\text{you land after 2.5 seconds.}}$ At that instant, your velocity is $h'(2.5) = -1.6 \cdot 2.5 = -4$, so you hit the ground at $\boxed{\text{4 meters per second}}$, which is about 14 kilometers per hour. So a 5 meter fall is pretty harmless on the moon.

Note. The analysis we've done above (determining a specific antiderivative using the general antiderivative plus one value of the function) is an example of what's called solving an "initial value problem," since it turns one single "initial" value into an entire function for all time.

5 Some sample antiderivatives

Here are a couple antiderivatives to find, to get some practice with finding them by guesswork. The last one is a challenge; we won't expect you to do anything like this for assignments or exams, but it may be fun for you to attempt (unless you've seen integration by parts before, in which case it is not difficult at all).

For each function, find **some** antiderivative (don't worry about the $+C$).

1. $x^2 + 7$
2. $x + \cos x$
3. e^{7x}
4. $\dfrac{x + x^7}{x^2}$
5. $x \cdot e^x$

Solutions.

1. We know that $\frac{1}{3}x^3$ is an antiderivative of x^2, and $7x$ is an antiderivative of 7, so adding them together gives a function whose derivative is $x^2 + 7$: $\boxed{\frac{1}{3}x^3 + 7x}$.
2. Again, this is a sum, so find an antiderivative for each of the terms individually and add: $\boxed{\frac{1}{2}x^2 + \sin x}$.
3. The derivative of e^{7x} is $7e^{7x}$, by the chain rule. So how can we get just e^{7x} as a derivative? Divide by seven: $\boxed{\frac{1}{7}e^{7x}}$.
4. This one is tough if you try to meet it head-on. But a quick re-expression makes it much easier: $\dfrac{x + x^7}{x^2} = \dfrac{1}{x} + x^5$. Now we can find an antiderivative for each term separately, to find $\boxed{\ln|x| + \frac{1}{6}x^6}$.
5. This one is tough (again, I emphasize: tougher than we'll expect you to do). Try a little experimentation: you know how to get e^x: it is the derivative of e^x. So what if you tried to get xe^x by just multiplying by x? It doesn't work: the derivative of xe^x is, by the product rule, $e^x + xe^x$. But actually, this *does* get you almost there: you just need to get rid of that extra e^x, which you know is the derivative of e^x. So in fact, xe^x is the derivative of $\boxed{xe^x - e^x}$, which is therefore an antiderivative.

This last example can be solved mechanically by a procedure called "integration by parts," which is covered in Math 1B. It is not as complicated as it sounds: it really is just a formalization of the sort of wishful thinking (and subsequent correction back to reality) that I've shown in the example above. In any case, you don't need to worry about this for now; I've included this as a had problem for students who have followed everything else easily.

6 Antiderivatives are hard

Here's a blanket warning: finding antiderivatives is **hard.** Whereas derivatives can often be computed by mechanically following rules (like the product rule, chain rule, etc.), sometimes to take an antiderivative requires very inspired guessing. Or worse: there may be no easy answer at all! For example, I promise that you will get an A in this class if you can guess this antiderivative (in closed form, in terms of simple functions, without any infinite sums or anything like that).

$$\int e^{-x^2} dx$$

We'll learn some good methods (and there are many more in Math 1B), but you should know that by and large, we are lucky if we can find an antiderivative. In many applications, computer approximations are used instead.