

Lecture 27: Definite integrals

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Only by admitting an infinitesimal unit for observation – a differential of history, that is, the uniform strivings of people – and attaining to the art of integrating them (taking the sums of those infinitesimal quantities) can we hope to comprehend the laws of history. Leo Tolstoy, *War and Peace* III.3.1

1 Introduction

Today we define the *definite integral*, which is nothing but a precise notation for the exact area under a curve. I will discuss two ways to conceive of what this means: one classical (from the first 200 or so years of the existence of calculus) and one modern (from the last 150 years or so).

We will compute a few definite integrals in cases where nothing is required but elementary geometry. We will then discuss the notion of *signed area*, which is a mild generalization of “area under a curve” that is more useful in most applications, and state some formal properties. The next lecture concerns the fundamental theorem that is used to actually compute definite integrals by hand.

The reference for today is Stewart §5.2.

2 The definite integral: notation

The following is the usual notation for what is called a “definite integral.”

$$\int_a^b f(x)dx$$

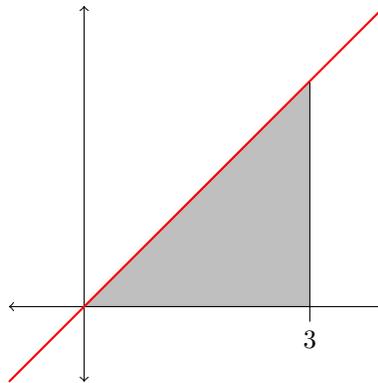
It means, roughly speaking: “the area under the curve $y = f(x)$ between $x = a$ and $x = b$.”

Caveat: this definition is a bit misleading when $f(x)$ has negative values. I’ll discuss this later, in the section “signed area.” For the time being, I’ll just consider positive functions so that this basic intuitive definition works.

For example, consider this expression.

$$\int_0^3 xdx$$

The graph $y = x$ is a straight line of slope 1. So this notation refers to the area shown.

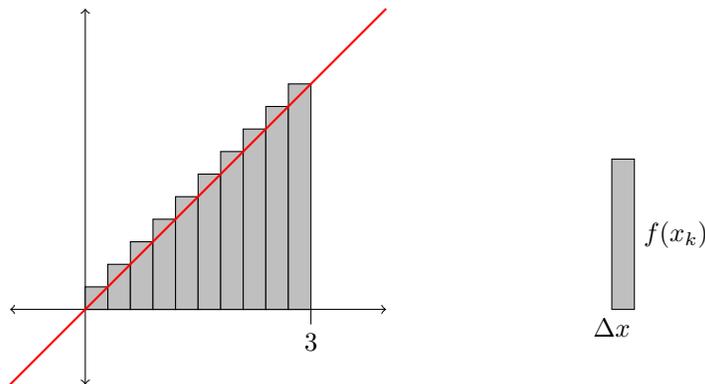


This is just a triangle with base and height 3; its area is $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} = 4.5$. Therefore

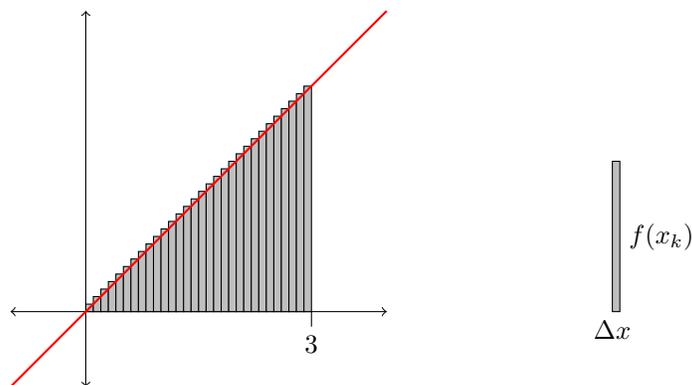
$$\int_0^3 x dx = 9/2$$

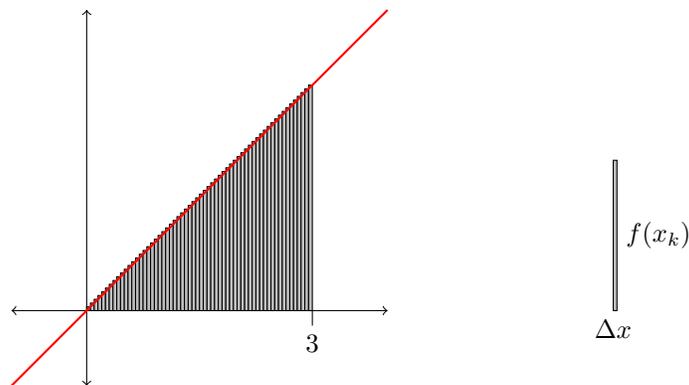
3 The classical viewpoint: sums of infinitesimal quantities

The classical viewpoint on the notation $\int_a^b f(x) dx$ explains why the notation is written the way it is. Remember that in the previous lecture, we saw that you could *approximate* the area under $f(x)$ from a to b by summing the areas of n rectangles, each of width Δx and height x_k (for some value x_k in the interval).

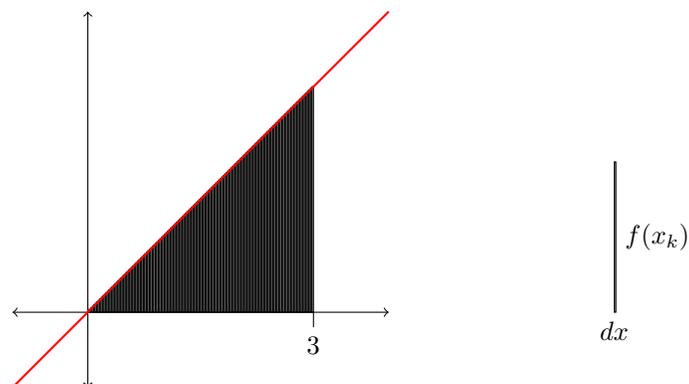


As the number of subdivisions becomes smaller and small, this approximation will become better and better.





The classical viewpoint is: every single one of these approximations is incorrect, because they all choose a unit (the width) totally arbitrarily. So instead, we should take an infinitely small (infinitesimal) width, written dx , and sum up infinitely many rectangles that are all infinitesimally thin.



Now, just as the notation for a sum of finitely many rectangles of some finite width is written this way,

$$\sum_{k=1}^n f(x_k)\Delta x$$

the classical viewpoint thinks of the following notation as meaning the sum of infinitely many rectangles, each of the infinitesimal width dx , for values of x from a to b .

$$\int_a^b f(x)dx$$

Note the similarity between these two notations: both have a symbol means to means sum (the greek letter S, Σ in the first case, and a stylized roman S in the second case).

According to the classical view, summing the infinitesimal rectangles will produce no error at all, because they eliminate the arbitrary choice of unit that would plague usual rectangular approximations.

The Tolstoy quote that began this lecture (which Tolstoy says in a passage where he speaking about historical analysis, but uses math as an example) speaks to this older viewpoint on integrals: they were formerly regarded as a sum of infinitely many quantities (differentials) that are all infinitely small. This is totally analogous to how the notion $\frac{dy}{dx}$, meant as an infinitesimal version of $\frac{\Delta y}{\Delta x}$, is meant to evoke arithmetic with infinitesimal quantities.

For this reason, the field we now call calculus was originally called “infinitesimal calculus,” roughly meaning “the practice of calculating with infinitesimal numbers.” The first textbook on calculus (written by L’Hôpital) was, in fact, titled “Analysis of the infinitely small.”

This classical perspective is still widely used colloquially, especially in physics. I think it is the best intuitive way to think about integrals. Nonetheless, it does invite various philosophical objections about whether it could possibly be sensible; it was mainly to answer these objections that the modern perspective (described below) emerged.

4 The modern view: limits

Recall that when we studied derivatives, we wanted to define the slope of the tangent line $\frac{dy}{dx}$ to be something like the slope of a secant line to an infinitely near point. To make this precise, however, we used the notion of limits: we said that for any finite distance Δx , we could calculate the slope of the secant line to a point Δx units away as $\frac{\Delta y}{\Delta x}$. These numbers actually settle down and approach a limit as Δx shrinks to 0. So rather than speaking of “infinitesimal differences,” we just spoke about limits of secant lines, and defined $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$.

The same perspective leads to the modern way of defining integrals. Instead of speaking directly about “a sum of infinitely many things,” we just speak of the sum of n things, and then take a limit as n grows to infinity. Thus, following Bernhard Riemann, we can make the following modern definition.

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{k=0}^n f(x_k)\Delta x$$

(where, for a chosen value of n , Δx stands for $(b - a)/n$ and x_k stands for $a + k\Delta x$).

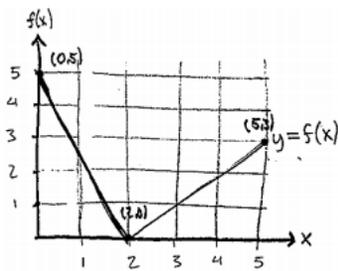
The modern view has two primary merits.

- **Philosophical:** this provides a clear, unambiguous definition of the notation $\int_a^b f(x)dx$ that avoids the ontological vagueness involved in “summing infinitesimal quantities.”
- **Computation:** Riemann sums can be computed explicitly with computers. Therefore we can use the modern view to approximate integrals that would be impossible to evaluate explicitly (all that remains is to answer the question: how large should n be for a good approximation? This sort of problem is the subject of a huge part of applied mathematics).

5 Some examples

However you think of the definite integral (as a sum of infinitesimal pieces, or as a limit of sums), you can always just think of it as a way to measure area. In some simple cases, integrals can be evaluated just by measuring area using basic geometry. Here are three examples.

Example 5.1. Consider the integral $\int_0^5 f(x)dx$, where $f(x)$ is the piecewise linear function shown below.

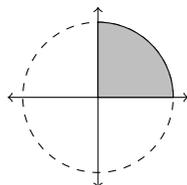


The area under this curve consists of two parts: the area under the segment from $(0, 5)$ to $(2, 0)$, and the area under the segment from $(2, 0)$ to $(5, 3)$. The first is a triangle of area $\frac{1}{2} \cdot 5 \cdot 2 = 5$, while the second is a triangle of area $\frac{1}{2} \cdot 3 \cdot 3 = 9/2$. So the total area under the curve is $5 + 9/2 = 19/2$ (or 9.5).

$$\int_0^5 f(x)dx = 19/2$$

Example 5.2. Consider $\int_0^1 \sqrt{1-x^2}dx$.

The graph of $y = \sqrt{1-x^2}$ is a semicircle of radius 1. To see this, notice that it just consists of all points on the curve $x^2 + y^2 = 1$ where $y \geq 0$. The area described by this integral will just be the area under this semicircle from $x = 0$ to $x = 1$, which is a quarter-circle, as shown.



The whole circle has area π . Therefore

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}.$$

Example 5.3. Let x be a variable. Consider the integral $\int_0^x tdt$.

If we graph the function $f(t) = t$, we will obtain a straight line of slope 1. So the area under this curve from 0 to x (where $x > 0$ for now) is the area of a right triangle with base x and height x . Therefore

$$\int_0^x tdt = \frac{1}{2}x^2$$

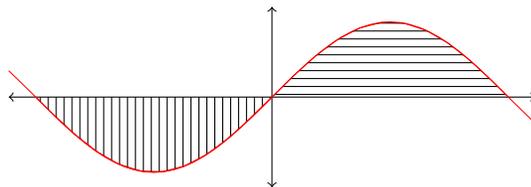
Notice that $\frac{1}{2}x^2$ is actually an antiderivative of $f(x) = x$. This is not a coincidence, and this is a topic will be at the center of the next lecture.

6 Signed area

In this section, we extend the meaning of the notation $\int_a^b f(x)dx$ to two new cases: when $f(x)$ could have negative values, and when $a > b$.

6.1 Negative values of $f(x)$

First, consider the case where $f(x)$ is sometimes negative. For example, consider the graph of $\sin x$ from $x = -\pi$ to $x = \pi$.



In this case, part of the graph is below the axis (vertical shading), and part of it is above the axis (horizontal shading).

In cases like this, we use the following convention.

Area below the x axis counts negatively towards the value of the definite integral.

Another way to put this is:

$$\int_a^b f(x)dx = (\text{area above}) - (\text{area below})$$

In fact, this is completely in line with what will happen if we compute using Riemann sums: when $f(x)$ is negative, we'll be adding terms $f(x_k)\Delta x$ to the sum, which will be negative. In case of $\sin x$, we can see that the part below is exactly the same shape as the part above, so without knowing anything else about this integral, we can conclude that

$$\int_{-\pi}^{\pi} \sin x dx = 0$$

This convention (area below the axis is subtracted) is also the best to use in most applications. For example:

Example 6.1. Suppose that $p(t)$ measures the profit per day of a company at a time t . If $p(t) > 0$, this means the company is making profit; if $p(t) < 0$ this means that it is losing money. Then

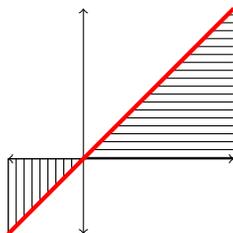
$$\int_0^{10} p(t)dt$$

perfectly captures the *total profit* of the company from time $t = 0$ to time $t = 10$: the positive parts add up the profit (on days when the company made money), and the negative parts add up the total loss (on days when the company lost money).

For these reasons, we sometimes say that the definite integral computes **signed area**: it counts some area as positive, some area as negative, and adds it all together.

Example 6.2. Compute $\int_{-1}^2 x dx$.

Solution. The graph of $y = x$ is shown, with the area between the graph and the x -axis shaded (vertical shading will be negative area, horizontal shading will be positive area).



The positive part is a triangle with area $\frac{1}{2} \cdot 2 \cdot 2 = 2$, while the negative part is a triangle with area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Therefore $\int_{-1}^2 x dx = 2 - \frac{1}{2} = 3/2$.

6.2 “Backwards” intervals

Next consider the situation where we write $\int_a^b f(x)dx$, but $a > b$. For example, what should we mean by $\int_2^0 x dx$? The answer is given by the following convention.

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

Therefore, we can always re-express an integral as one whose endpoints a and b satisfy $a < b$. For example: $\int_2^0 x dx = -\int_0^2 x dx$, which is -2 because $\int_0^2 x dx$ is the area of a triangle with base 2 and height 2. What is the reason for this convention? There are two good reasons for it.

- **Theoretical.** You can think of $\int_a^b f(x) dx$ as being a sum of infinitesimal quantities $f(x) dx$, where x is moving from a to b , and dx measures its change. Now, if a is actually greater than b , this means that x is moving *backwards*, so dx will be *negative*. So the resulting sum is the same as if x moved forward from b to a , except that everything is made negative. Alternatively, if you use the definition of Riemann sums, then if $b < a$ then $\Delta x = \frac{b-a}{n}$ will be negative, and so the Riemann sum will end up counting everything with a minus sign.
- **Practical.** This convention keeps many identities and computational techniques about integrals neat and tidy.

To explain what I mean by neat and tidy, consider for example the following identity.

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

This identity is visually obvious if $a < b < c$: we're computing one area by breaking it into two pieces (the area above $[a, b]$ and the area above $[b, c]$). By taking the sign convention above, this identity is *universally true*. For example, if $a < c < b$, then identity above says that to compute the area above $[a, c]$, we can compute the area above $[a, b]$ and then *subtract* the area above $[c, b]$.

Identities like this are very important in computations, so it is very useful to make sure that they are universally true, so that you don't have to worry about strange edge cases when applying them.