Lecture 5: Continuity II

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1 Introduction

Today we continue our discussion of continuity. We begin by stating a seemingly obvious feature of continuous functions, but one which is conceptually important: the intermediate value theorem. The way I look at this theorem is it is just one more way to wrap your head around what the word "continuity" means and what it doesn't. The theorem basically says that a continuous function never teleports – it cannot get to the other side of a barrier without crossing that barrier.

We will then discuss some of the main types of discontinuity, which you can regard as different ways to teleport (and thus break the intermediate value theorem). We'll conclude with a discussion of various ways that you can conclude that a function is continuous.

The reference for this lecture is Stewart §2.4 (same as the previous).

2 The intermediate value theorem

Suppose that one day, you are approached by a mugger, who demands to know the first few decimal places of the square root of 2. Your hands had better stay where he can see them, so no reaching for a calculator! Sensing that you're nervous, and a little desperate himself, he tells you he'll settle for the first digit. Well that's easy, isn't it? It can't be any bigger than 2, since $2^2 = 4$, but it can't be less than 1 since $1^2 = 1$, so it the first digit must be 1.

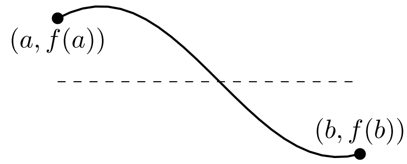
Maybe he wants at least the tens digit, though. Now you have to think a bit harder. You know it's somewhere between 1 and 2, so what about 1.5? If your mugger kindly offers you some chalk to write on the brick wall behind you (hands in the air the whole time, though!) you can work out that this is $1.5^2 = 2.25$. So it's too big! What about 1.4? As long as the chalk doesn't slip from your not rather sweaty fingers, you can find that $1.4^2 = 1.96$. So, you assert confidently, the tenths digit is 4, because 1.4 is too small and 1.5 is too big. So the square root of 2 is somewhere win the middle.

Unfortunately for you, this is a rather paranoid mugger. *Why*, he demands to know, *should I trust you that* $\sqrt{2}$ *between* 1.4 *and* 1.5*? Just because* 2 *is between* 1.4^2 *and* 1.5^2? Perhaps at this point you should make a break for it, because this mugger is clearly beyond reason if he would doubt something so simple as what you've just shown him.

Another option is to remind him of the following theorem.

Theorem 2.1 (Intermediate value theorem)**.** *If $f(x)$ is a function that is continuous for all x in $[a, b]$ (where a and b are any two numbers), then it attains every value between $f(a)$ and $f(b)$ for some value of x between a and b.*

This theorem is pretty abstract as I've stated it above, but visually it is much simpler to understand. It just means that if $f(x)$ is continuous, and part of its graph starts on one side of a horizontal line, and ends on the other side, then somewhere in the middle it actually *touched* the horizontal line.



So, you could confidently assure your mugger: the function $f(x) = x^2$ is continuous, so if $f(1.4) < 2 < f(1.5)$, then there is *guaranteed to be some value* x between 1.4 and 1.5 such that $f(x) = 2$. That is, there is a number between 1.4 and 1.5 such that $x^2 = 2$. This is of course the square root of 2, and its decimal expansion does indeed begin with 1.4.

Here are a couple more problems that can be solved using the intermediate value theorem.

Example 2.2. Is there a number x such that $x^4 + x = 7$?
Solution. Yes. Let $f(x)$ be the function $f(x) = x^4 + x$. Then $f(0) = 0$ and $f(2) = 18$. Some for some c between 0 and 2, $f(c) = 7$, by the intermediate value theorem.

Example 2.3. Has your height (in inches) ever been equal to your weight (in pounds)?
Solution. Yes. When you were born, your height (in inches) was almost certainly greater than your weight (in pounds); your height was probably at least 12 inches, while your weight was probably at most 12 pounds. By now, however, this situation has almost certainly reversed (for example, I am 74 inches tall and I have no shame in saying that I am over 74 pounds). So if you consider the function $d(t) = h(t) - w(t)$, where $h(t)$ is your height after t years (in inches) and $w(t)$ is your weight (in pounds), then $d(0)$ was positive, while $d(t)$ is now negative. By the intermediate value theorem, there was some t during your lifetime when $d(t)$ was exactly 0. This is assuming, of course, that you height and weight are continuous in time, which is a pretty safe assumption barring traumatic circumstances.

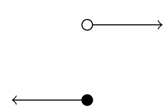
Example 2.4. A monk begins hiking one morning and reaches the top of a mountain. The following morning, he hikes down. Was there any time during the second day when the monk was at exactly the same place on the mountain as he was exactly 24 hours ago on the hike up the mountain?
Solution. Yes. Let $u(t)$ be his distance up the mountain at t o'clock during the hike up (where by "t o'clock" mean for example 10:30am when $t = 10.5$ and 3:12pm when $t = 15.2$). Let $d(t)$ be his distance up the mountain at time t on the hike down on the second day.

We know that $d(0) - u(0)$ is positive (since he begins at the top the second day and the bottom the first day), but $d(24) - u(24)$ is negative (for similar reasons). So assuming these functions are continuous (no teleporting monks!), then their difference is also continuous, and by the intermediate value theorem, there is some t such that $d(t) - u(t) = 0$, i.e. $d(t) = u(t)$.

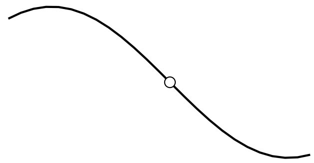
3 Types of discontinuity

Some types of discontinuity are frequent enough (at least in the little playground of functions we inhabit in a class like this) that they have special names. I mention these types now because they highlight one conceptual use of a statement like the intermediate value theorem: different sorts of discontinuity often vary essentially in how they break the intermediate value theorem.

A **jump discontinuity** occurs when the function is continuous on the left and the right, but the two one-sided limits are not equal; the function essentially teleports from one place to another. An example of such a function is the amount of money you have been paid by an employer, given that you are paid once a week.

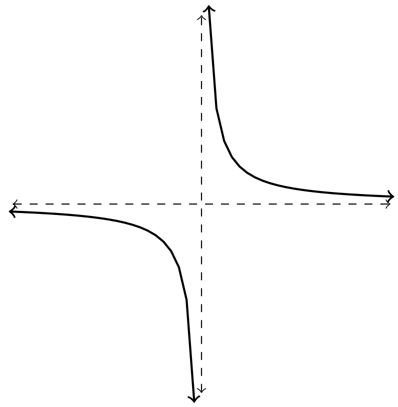


A **removable discontinuity** is a value where the limit exists but the value of the function is either undefined or not equal to the limit. I kind of think of this discontinuity as a "playground argument" discontinuity, because it sounds like the sort of retort that a kindergartner would use in an argument ("oh yeah? Well, the functions *isn't defined there* so it doesn't cross the line after all!").



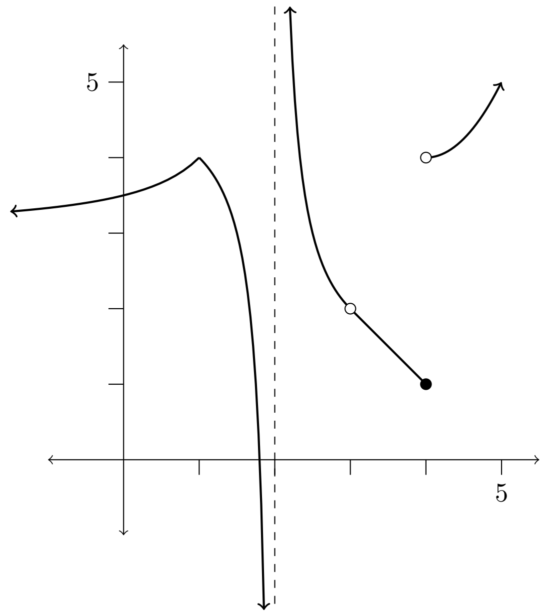
An example of a function with such a discontinuity is $f(x) = \frac{x^2-1}{x-1}$ (the slope of a secant line through $(1, 1)$ and (x, x^2)). Another example is $f(x) = \frac{\sin x}{x}$. Both have discontinuities because you would have to divide 0 by 0 to define them at some point.

A **vertical asymptote** is a value where the function blasts off to infinity on both sides to escape its obligations to be defined there. I like to think of this as a "Pacman discontinuity," since it resembles the feature in the classic Pacman game where the player can leave one side of the board and reappear on the other. The basic example for this is $f(x) = \frac{1}{x}$ at $x = 0$.



To summarize: the three main types of discontinuity that we've met in this course differ essential in the way in which the intermediate value theorem fails. The theorem says essentially that a graph can start on one side of a horizontal line and finish on the other. We've seen three ways it could fail to do this: it could jump right over the line (jump discontinuity), it could apparently cross it but yet be undefined (have a little "open circle" in our notation) at the very instant it would cross the line, or it can blast off to infinity and come back on the other side (vertical asymptote).

Example 3.1. Identify all of the discontinuities of the following graph, with their types.



Solution. There is a vertical asymptote at $x = 2$, a removable discontinuity at $x = 3$, and a jump discontinuity at $x = 4$. The "corner" at $x = 1$ is not a discontinuity, but it is still an interesting type of non-typical behavior; we will see later in the course that it corresponds to a point where the derivative of the function is discontinuous.

4 Identifying continuous functions

One of the most useful aspects of continuous functions is that it is very easy to take their limits: their limits are just their values. This feature makes them very useful in the squeeze theorem, where we can compute complicated limits by computing easier limits, often as values of continuous functions. For purposes like this, it is useful to have as complete a catalog as possible of continuous functions. Most functions that you think should be continuous are continuous; **this list is NOT exhaustive.** But I will highlight some of the main ways to conclude that a function is continuous.

The most basic functions that we encounter are polynomials. These are functions like the following.

- $f(x) = x + 1$
- $f(x) = x^2$
- $f(x) = 2x^7 + 852x^5 + 7x + 3$

A basic fact that **polynomial functions are continuous everywhere.** Some texts prove this using techniques like "limit laws" (for example, in Stewart).

Slightly more complex are *rational functions*. These are functions like the following.

- $f(x) = \frac{x+1}{x+2}$
- $f(x) = \frac{1}{x}$
- $f(x) = \frac{x^2+1}{x^3-3x^2+3x-1}$

Rational functions have slightly more exotic graphs – they usually have a couple vertical asymptotes, and often other sorts of asymptotes as well. Nonetheless, they are well-behaved away from their vertical asymptotes. **Rational functions are continuous wherever their denominator is nonzero.**

The **following functions are continuous everywhere.**

- Basic trigonometric functions $\sin x, \cos x$.
- Exponential functions e^x and more generally a^x for any (positive) base a.

Most of the other functions you've seen in precalculus are not necessarily continuous everywhere, but they are at least continuous where they are defined. For example, the functions $\ln x$, any other logarithmic function, the inverse trigonometric functions, and "radical" functions like $\sqrt{x}$, $\sqrt[3]{x}$ and so forth are continuous wherever they are defined.

There are also some rules you can use to conclude that more complicated functions are continuous.

A sum or product of continuous functions is continuous.

Example 4.1. The functions $f(x) = x^{27}$ and $g(x) = \cos(x)$ are continuous. Therefore the function $x^{27}+\cos(x)$ is also continuous, as is the function $x^{27} \cdot \cos(x)$.

A quotient of two continuous functions is continuous wherever the denominator is nonzero.

Example 4.2. The functions $\sin x$ and $\cos x$ are continuous, and $\cos x$ is zero only at odd multiples of π (that is, $\pm\pi, \pm 3\pi, \pm 5\pi$, and so forth). Therefore the function $\tan x = \frac{\sin x}{\cos x}$ is continuous at all values except odd multiples of π.

A composition of two continuous functions is continuous. Here a "composition" means plugging one function into the other; it is easiest to make clear what I mean with a few examples.

Example 4.3. The function $\sqrt{x}$ is continuous except when x is negative (where it is not defined, at least for purposes of this class[1]). The function $x^2 + 1$ is continuous and always positive. So the function $\sqrt{x^2+1}$ is continuous everywhere.

Example 4.4. The function $e^{\sin x}$ is continuous everywhere, since e^x and $\sin x$ are continuous everywhere.

[1] I do not mean to disparage imaginary numbers. I practically make my living thinking about imaginary numbers! But let's not think about them for the time being.