Lecture 9: Differentiating polynomials and exponential functions

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NOTE: This material will **not be covered on the first midterm.**

1 Introuction

Today we will begin to write down some shortcuts for computing derivative of some common sorts of functions. The first functions we consider will be polynomials and exponential functions. Polynomials are the most basic functions in mathematics, for two reasons: first, they are the functions that are "built up" from only addition and multiplication. Second, they are the only functions which become 0 after being differentiating sufficiently many times. Exponential functions are particularly simple in a different way: they very closely resemble their own derivatives. This is the main reason exponentials are so important in many areas of pure and applied mathematics. We will finish by remarking why mathematicians so often insist on representing all exponentials using the constant called e: the function e^x has the very convenient feature of being its own derivative.

The reference for today is Stewart §3.1.

2 The derivative of x^n

From now on, we will freely use the following fact, without bothering to evaluate any limits. This is called the **power rule**.

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}} \quad \text{(where } n \text{ is a constant)}$$

Warning. It is very important to remember that this rule is only valid when n is a *constant*, and x is the variable. For example, a very common mistake is for students to attempt to differentiable $f(x) = 2^x$ and obtaining $x \cdot 2^{x-1}$, which is not true (for example, this would be negative for $x = -1$, even though $f(x)$ is an increasing function everywhere).

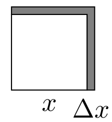
For example:

- For any value of x, $x^0 = 1$. So this rule says that $\frac{d}{dx}1 = 0$. This is just the fact that a **constant function has a zero derivative**.
- For $n = 1$, this rules says that $\frac{d}{dx}x = 1$. This is not too surprising, since the notation really suggests that $\frac{dx}{dx}$ ought to be 1. This is just the fact that **the derivative of a linear function is constant, equal to the slope of the graph.**
- For n=2, we get the fact that $\frac{d}{dx}x^2 = 2x$, which we've seen a couple times before.

You can simply memorize this rule if you like, but I will mention a few mnemonics for remembering where it comes from. These are not part of the course; I include them in case they are helpful for you to understand the fact.

2.1 A visual mnemonic

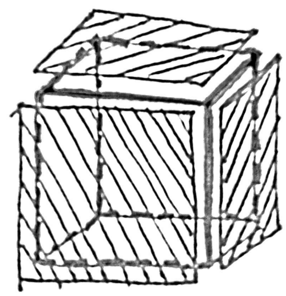
Think of x^2 as the area of a square with side length x. Now imagine that you change x be a very small amount, Δx. This changes the square as follows.



$$\Delta(x^2) \approx 2x\Delta x$$

If Δx is very small, the change in the area of the square just consists of this little stripe, of width Δx. This stripe is essentially two pieces, each of length x and width Δx. So this suggests that $\Delta x^2 \approx 2x\Delta x$. (In fact, Δx^2 is exactly $2\Delta x + (\Delta x)^2$; the second term becomes negligible in the limit).

A similar picture results when you imagine x^3 as the volume of a cube with side length x, although it is somewhat tricky to draw on a page. Here is a rough sketch.



The cube grows out in three directions. Each direction gives one slice, of thickness Δx and area x^2. So it appears from this picture that $\Delta(x^3) \approx 3x\Delta x$.

It takes a bit of imagination, but you can convincr yourself that the same picture should be true in 4 or more dimensions also – an n-dimensional "hypercube"[1] grows outward in n directions, each of which leads to a $(n-1)$-dimensional hypercube. Thus $\frac{d}{dx}x^n = nx^{n-1}$.

2.2 Expanding $(x+h)^n$

Some of you may know the trick where you can use Pascal's triangle to expand the expression $(x+h)^n$. This is one way to get the power rule. Actually, though, one basic insight is that you don't need to expand the whole thing: just the first two terms will suffice. Just group anything that is multiplied by h^2 together and don't bother to computer what exactly it is. Notice that:

[1]This is the real word used by mathematicians.

$$\begin{aligned}
(x+h)^2 &= x^2+2xh+h^2 \\
(x+h)^3 &= (x+h)(x+h)^2 \\
&= (x^3+2x^2h+h^2(\cdots))+(x^2h+h^2(\cdots)) \\
&= x^3+3x^2h+h^2(\cdots) \\
(x+h)^4 &= (x+h)(x+h)^3 \\
&= (x^4+3x^3h+h^2(\cdots))+(x^3h+h^2(\cdots)) \\
&= x^4+4x^3h+h^2(\cdots)
\end{aligned}$$

This pattern continues. For any positive integer n, expanding $(x+h)^n$ yields $x^n + nx^{n-1}h + h^2(\cdots)$ (where I don't care what all has been shuffled together into this $\cdots$ symbol). The result of this is that the slope of secant line to the graph $y = x^n$ has slope $\dfrac{(x+h)^n - x^n}{h} = \dfrac{nx^{n-1}h + h^2(\cdots)}{h}$, or in other words $nx^{n-1} + h(\cdots)$. The limit as $h \to 0$ is therefore just nx^{n-1} – everything grouped together under the $\cdots$ disappears because it is multiplied by h, which becomes 0.

3 Differentiating polynomials in general

All that is needed to differentiate any polynomial are the following two basic rules. In words: the derivative of a sum is the sum of the derivatives, and the derivative of a constant multiple is the same constant multiple of the derivative. In symbols, these rules are:

1. $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$.
2. For C a constant, $\frac{d}{dx}(Cf) = C\frac{d}{dx}f$.

Warning. It is very important that C is a constant here. If it is also a function of x, the correct method to use is called the product rule, which we'll see later.

This two rules, with the power rule, make it simple to evaluate the derivative of any polynomial. For example:

$$\begin{aligned}
\frac{d}{dx}(7x^4+4x^2+9x+2) &= \frac{d}{dx}(7x^4)+\frac{d}{dx}(4x^2)+\frac{d}{dx}(9x)+\frac{d}{dx}(2) \\
&= 7\frac{d}{dx}x^4+4\frac{d}{dx}x^2+9\frac{d}{dx}(x)+2\frac{d}{dx}(1) \\
&= 7\cdot 4x^3+4\cdot 2x+9\cdot 1\cdot 1+2\cdot 0 \\
&= 28x^3+8x+9
\end{aligned}$$

I've only shown these steps in full detail to emphasize that all we are using are the power rule and the rules for sums and multiples. You are free to skip straight to the last line in your work (or to the second to last if the multiplication is a tricky).

4 Differentiating exponential functions

We can't quite differentiate exponential functions immediately from the limit definition of derivative, but the limit definition still shows something fairly remarkable.

For example, suppose that $f(x) = 2^x$. Let's see what happens if we attempt to compute the derivative at an arbitary value $x = c$.

$$
\begin{aligned}
f'(c) &= \lim_{h \to 0} \frac{2^{c+h} - 2^c}{h} \\
&= \lim_{h \to 0} \frac{2^c \cdot 2^h - 2^c}{h} \\
&= \lim_{h \to 0} 2^c \frac{2^h - 1}{h} \\
&= 2^c \lim_{h \to 0} \frac{2^h - 1}{h} \\
&= 2^c \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\
&= 2^c \cdot f'(0)
\end{aligned}
$$

So what falls out of this is that to compute the derivative at *any* point, it's enough to compute just the value $f'(0)$.

The result of this could also be written in terms of th derivative function, like this.

$$f'(x) = f'(0) \cdot f(x)$$

This statement is true for any exponential function at all – if you look at how wr obtained it, you will see that it didn't matter that we were using 2^x, rather than 3^x or e^x. Every one of these functions has a common feature: the derivative is just a constant multiple of the original.

We won't actually write down what, precisely, this constant multiple is today. I will only remark that there is only one choice where the constant is exactly one, i.e. one function $f(x) = c^x$ that is equal to its own derivative. That function is e^x. This is a very important fact; probably it is the most important fact about the number e, and the main reason that it is so common in mathematics.

$$\boxed{\tfrac{d}{dx} e^x = e^x}$$

You have probably wondered why we call the logarithm base e the "natural logarithm" (and give it its own button on calculators), or why so many exponential functions are expressed in terms of e^x. Essentially, it is the same reason that your science classes insist on using the metric system (centimeters, kilometers, etc.) rather than the more familiar (in the US) imperial units (inches, miles, etc.): the metric system makes unit conversions much easier and less error prone. Similarly, expressing exponentials and logs in terms of the number e makes differentiation less error-prone. It is the most reasonable benchmark to use in calculus.

As a final remark, nothing is lost by only thinking about e^x, because other exponentials can be expressed with it. For example, notice that the function 2^x can also be written $e^{\ln 2x}$. The latter looks more complex, but it is actually much easier to work with in practice.