Lecture 20: Asymptotics

Nathan Pflueger

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1 Introduction

As we have seen, whether or not a series converges does not depend on the detailed behavior of the terms, but only on their general rate of decay. We have frequently made use of informal reasoning about the general shape of terms of a series to make an initial conjecture about whether it converges or not. Establishing the truth of these conjectures was usually done by the comparison test, or some other test.

In this lecture, we consider some methods to make this type of reasoning precise. This sort of analysis is usually called *asymptotics*. The goal is to take some series, and "simplify" it by devising another series that is different, but has the same sort of long-term behavior. This is called *asymptotic simplification.* This is useful because asymptotic simplification, while it may significantly alter the series, does not alter whether or not it converges; this makes it a convenient way to study the convergence or divergence of complicated series.

There are two main bits of notation when studying asymptotics: we will denote by $a_n \sim b_n$ the statement that "a_n resembles b_n eventually," and we will denote by $a_n \ll b_n$ the statement that "a_n grows more slowly than b_n eventually." These will be given precise definitions in section 3. Intuitively, you should think of $\sim$ as corresponding to an asymptotic version of $=$, and $\ll$ as corresponding to an asymptotic version of $<$.[1]

One of the main benefits of studying asymptotics is that it develops the habit of always having a "back of the envelope" understanding of functions under consideration. In the real world, you will frequently not need (or have the time to) actually do careful computations; what is most important is having a general sense for shape.

The reading for today is the handout titled "Asymptotics," listed under "Reading for the course." The homework is problem set 19 (which includes weekly problems 18 and 19) and a topic outline. You should begin working on weekly problems 23 and 24.

Due to the midterm tomorrow evening, the homework assigned today is not due until Friday. Note that the homework assigned on Wednesday will also be due on Friday as originally planned.

2 Examples of asymptotic simplification

Here are some examples from recent homework assignments, illustrating the sort of simplification that we wish to be able to do.

The purpose of these examples is to illustrate the ways in which we want to be able to use the symbols $\sim$ and $\ll$. In particular, $a_n \ll b_n$ should mean that a_n can safely be neglected compared to b_n, and $a_n \sim b_n$ should mean that a_n and b_n can safely be taken interchangeably when considering long-term behavior.

These examples should fairly intuitively illustrate the sort of thing that these symbols should mean. The next section will give precise definitions that will justify this reasoning.

[1] The symbol $\ll$ is also sometimes used informally to mean "much less than," but for our purposed it will mean "asymptotically less than."

Example 2.1 (PSet 16, problem 4(c)). Consider the series $\sum_{k=1}^{\infty} \frac{1}{k+5}$. Since $k \gg 5$, we can safely neglect the 5 term.

$$\frac{1}{k+5} \sim \frac{1}{k}$$

Thus since the harmonic series diverges, this series should diverge as well. Note that in this case, the direct comparison test worked well. It would have worked less well, though, for $\sum_{k=6}^{\infty} \frac{1}{k-5}$, although the asymptotic comparison test works just as well on this series as on the given series.

Example 2.2 (PSet 16, problem 1(e) and PSet 17, problem 2(i)). Consider the series $\sum_{k=2}^{\infty} \frac{2^k+1}{3^k+1}$. Since $2^k \gg 1$ and $3^k \gg 1$, to study the long-term behavior, we can safely ignore the 1 terms.

$$\frac{2^k+1}{3^k+1} \sim \frac{2^k}{3^k}$$

Now, $\sum \frac{2^k}{3^k}$ is a geometric series with ratio less than 1, so it converges. Thus we expect the original series to converge as well (indeed, it does, as the asymptotic comparison test will show). Note that it is not true that $\frac{2^k+1}{3^k+1} \leq \frac{2^k}{3^k}$, so we can't quite apply the comparison test in the most naive way; asymptotic comparison helps a lot here.

Example 2.3 (PSet 18, problem 2(b)). Consider the series $\sum_{k=1}^{\infty} \frac{2k+1}{k^{3/2}}$. Since $2k \gg 1$, this simplifies as follows.

$$\frac{2k+1}{k^{3/2}} \sim \frac{2k}{k^{3/2}} = \frac{2}{k^{1/2}}$$

Since this is a p-series with $p < 1$, it diverges. So we expect the given series to diverge as well (again, the asymptotic comparison test will show that it diverges).

Example 2.4 (PSet 18, problem 2(c)). Consider the series $\sum_{k=10}^{\infty} \frac{5}{4^k+7}$. Since $4^k \gg 1$, this simplifies as follows.

$$\frac{5}{4^k+7} \sim \frac{5}{4^k}$$

Now, a geometric series with common ratio $\frac{1}{4}$ converges. The asymptotic comparison test will show that the given series converges.

3 Definition of $\sim$ and $\ll$

The definition of the notation $a_n \sim b_n$ is designed to mean that $a_n \approx b_n$ for large n, where the fit gets better and better as n gets larger. Another way to say this is that $\frac{a_n}{b_n} \approx 1$, where the approximation gets better and better as n gets larger. Fortunately, this last statement is something that we have a mathematical description of: the notion of a limit. We use it to make the definition.

Definition 3.1. The notation $a_n \sim b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. In this case, we shall say that a_n and b_n are *asymptotic.*

Similarly, we want the notation $a_n \ll b_n$ to mean, roughly, that a_n can be neglected compared to b_n, as n becomes large. One way to say this is that the ratio $\frac{a_n}{b_n}$ should be very close to 0 for large n, and get closer and closer as n grows to infinity. This, again, can be formalized using the language of limits.

Definition 3.2. The notation $a_n \ll b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. In this case, we shall say that b_n *grows faster than* a_n.

Observe that there is another way to thinking of the notation $\ll$. It should mean that we are allowed to neglect a_n when compared with b_n. In fact, it is not hard to show that the following is true (and could be used as a definition of $\ll$ if desired).

Fact. For two sequences a_n and b_n, b_n grows faster than a_n if and only if $a_n + b_n$ is asymptotic to b_n. In symbols, $a_n \ll b_n$ if and only if $a_n + b_n \sim b_n$.

4 Asymptotic comparison of some common functions

In order to do asymptotic simplification, we need to understand which functions grow faster than others. This section summarizes some main facts about various families of functions.

Note that in this section, I will be considering asymptotics of functions $f(x)$ rather than sequences a_n. The definitions are the same, as are the sorts of arguments.

First consider polynomials. Any polynomial is asymptotic to its leading term, as in the following examples.

$$\begin{aligned} x^2 + x + 1 &\sim x^2 \\ 5x^3 + 1000x^2 + 1 &\sim 5x^3 \\ x + 7 &\sim x \end{aligned}$$

This works because for any two exponent p_1, p_2, we have $x^{p_1} \ll x^{p_2}$ if and only if $p_1 < p_2$. So terms with smaller exponents can safely be ignored.

Next, consider exponential functions. Observe that e^x grows faster than any polynomial. This can be seen by consider the Taylor series of x, by by applying l'Hôpital's rule repeatedly, as in the following example.

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2+2x+3}{e^x} &= \lim_{x\to\infty} \frac{2x+2x}{e^x} \\ &= \lim_{x\to\infty} \frac{2}{e^x} \\ &= 0 \end{aligned}$$

This computation, which applies l'Hôpital's rule twice in a row, shows that $x^2 + 3x + 3 \ll e^x$. In fact, the same technique will work to show that any polynomial grows less quickly than e^x.

What about comparing different exponential functions, say comparing e^x to 2^x? In this case, the limit as $x \to \infty$ of $2^x/e^x = (2/e)^x$ is 0, since $2 < e$. This shows that $2^x \ll e^x$. The same sort of reasoning shows that $a^x \ll b^x$ if and only if $a < b$.

What about logarithms? In fact, logarithms grow more slowly than any polynomial. This, again, can be seen using l'Hôpital's rule. For example:

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{x^2+1} &= \lim_{x\to\infty} \frac{1/x}{2x} \\ &= \lim_{x\to\infty} \frac{1}{2x^2} \\ &= 0 \end{aligned}$$

It follows from this that $\ln x \ll x^2 + 1$. The same sort of argument would show that $\ln x \ll p(x)$ for any polynomial $p(x)$.

To illustrate how to use facts like these, consider the following example.

Example 4.1. Consider the function $f(x) = \frac{e^x+x^2}{\ln x+x^2}$. Because $x^2 \ll e^x$ and $\ln x \ll x^2$, we can safely neglect the x^2 in the numerator and the $\ln x$ in the denominator. Therefore we obtain the following asymptotic simplification.

$$\frac{e^x + x^2}{\ln x + x^2} \sim \frac{e^x}{x^2}.$$

This is a simplification, since this new function is somewhat easier to study.

5 The asymptotic comparison test

The main benefit of asymptotic analysis and asymptotic simplification, for our purposes, is that it makes convergence questions much easier to study. Two asymptotic sequences have the same convergence properties. This is expressed in the following.

Theorem 5.1 (Asymptotic comparison test)**.** *Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are two series such that* $a_n \sim b_n$ *and such that all terms* b_n *are nonnegative*[2]. *Then either both series converge or both series diverge.*

The restriction that $b_n \geq 0$ is necessary because, behind the scenes, the asymptotic comparison test invokes the direct comparison test.

Notice that the asymptotic comparison test is, in some sense, just a restatement of the limit comparison test stated several lectures ago.

There are a couple other, similar, statements that are sometimes included in the asymptotic comparison tests, which may sometimes be useful. These are stated below.

- If $a_n \ll b_n$, all $b_n \geq 0$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $a_n \ll b_n$, all $a_n \geq 0$, and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.

All three of these statements are essentially just souped-up versions of the direct comparison test.

6 Examples

Here are some examples of asymptotic simplification and asymptotic comparison in action.

Example 6.1. Does the series $\sum_{n=1}^{\infty} \frac{(n^2+1)(3n^3+n)}{(n^2+1)^2(7n^3+n^2)}$ converge?

First, asymptotically simplify the numerator and the denominator.

$$\begin{aligned} (n^2+1)(3n^3+n) &\sim n^2 \cdot 3n^3 = 3n^5 \\ (n^2+1)^2(7n^3+n^2) &\sim (n^2)^2 \cdot 7n^3 = 7n^7 \end{aligned}$$

From this it follows that

$$\frac{(n^2+1)(3n^3+n)}{(n^2+1)^2(7n^3+n^2)} \sim \frac{3n^5}{7n^7} = \frac{3}{7} \cdot \frac{1}{n^2}$$

Now, since $\sum \frac{1}{n^2}$ is a p-series with $p > 1$, it converges. So in fact the series in question converges, by the asymptotic comparison test.

[2]Actually, it suffices to only consider terms b_n for n sufficiently large.

Example 6.2. Does the series $\sum_{n=1}^{\infty} \frac{e^{-n}+n}{e^{-n}+n^2}$ converge or diverge?

Again, we can asymptotically simplify this. Using the fact that $e^{-x} \ll n$ and $e^{-x} \ll n^2$, the following holds.

$$\frac{e^{-n}+n}{e^{-n}+n^2} \sim \frac{n}{n^2} = \frac{1}{n}$$

Since the harmonic series diverges, this series also diverges by the asymptotic comparison test.

Note. Do not confuse e^{-n} with e^n. While e^n grows faster than any polynomial, e^{-n} goes to 0 as $n \to \infty$. In particular, it grows more slowly than any function that does not go to 0 as n goes to infinity.