Lecture 25: Differential Equations

Nathan Pflueger

4 November 2011

1 Introduction

A differential equation is an equation which involves a function and its derivatives. Many physical situations are most readily described in terms of differential equations; a prominent example is Newton's description of gravitation, which was one of the driving forces of the development of calculus as a whole. Other situations which lend themselves to modeling with differential equations involve the growth or decline of animal and plant populations over time, the spread of diseases, and the prices of financial instruments. What all of these situations have in common is that a particular number or collection of numbers is evolving in some way over time; it is the nature of this change that we hope to describe and understand with differential equations.

The study of differential equations revolves around three tasks: modeling, qualitative analysis, and explicit solution. The goal of modeling is to determine how a situation can be described with a differential equation (or system of differential equations). One of the main benefits of modeling is that it allows analogies to be made between different situations that are modeled by the same abstract equation. Qualitative analysis concerns drawing conclusions from a differential equation without solving it explicitly (for example, if the equation models a population, you might want to know whether the population will go to 0 over time or not, without wanting to know the exact population at different times). Finally, in some situations it is possible to explicitly solve differential equations. We will consider some cases of the most easily solved differential equations in this class.

This unit will begin with three lectures on modeling and qualitative analysis, followed by four lectures on explicit solutions to differential equations, and conclude with three lectures on systems of differential equations (that is, collections of interrelated equations). We will follow chapter 31 of Gottlieb's textbook fairly closely, although we will reverse the order of §31.4 and §31.5.

The reading for today is Gottlieb §31.1. The homework is problem set 24 and a topic outline.

2 What is a differential equation?

A differential equation is an equation involving a function and its derivative. For example, here is a simple differential equation.

$$f'(x) = f(x)$$

What this equation asserts is that f is a function that is equal to its own derivative. A *solution* to this equation is a function $f(x)$ that satisfies the equation. For example, $f(x) = e^x$ and $f(x) = 2e^x$ are both solutions of this equation. In fact, the only solutions to this equation are functions of the form $f(x) = Ce^x$, where C is a constant.

It is worth emphasizing that *the solutions of a differential equation are functions.* This is in contrast with, say, the equation $x^2 - 3x + 2 = 0$, where the solutions are the numbers $x = 1$ and $x = 2$. It is also worth emphasizing that *a differential equation will often have infinitely many solutions.*

To illustrate further the notion of a differential equation further, here are some more differential equations, together with some examples of solutions of them. Each of these represents an equation that we will eventually be able to solve in this course. I have mentioned them here so that they will be slightly familiar, and also to introduce how these equations are often written.

Equation	Solutions
$f'(x) = f(x)$	$f(x) = e^x$ $f(x) = 3e^x$
$y'' = 0$	$y = x + 1$ $y = 0$
$y'' = -y$	$y = \sin x$ $y = \cos x$ $y = \sin x + 2\cos x$
$M'(t) = 5 - 3M(t)$	$M(t) = \frac{5}{3} + e^{-3x}$ $M(t) = \frac{5}{3} - e^{-3x}$
$g''(t) - g'(t) - 2g(t) = 0$	$g(t) = e^{2t}$ $g(t) = e^{-t}$

While differential equations usually have infinitely many solutions, often they are accompanied by what are called *initial conditions*, which are information about the value of the solution function at a single point.

For example, suppose that I am looking for a function $f(x)$ such that $f'(x) = f(x)$. There are many such functions, but suppose that I also want it to satisfy $f(0) = 1$. In fact, there is only one such function: $f(x) = e^x$. The equation $f(0) = 1$ is called an initial condition. To summarize this example:

Differential equation: $f'(x) = f(x)$
Initial condition: $f(0) = 1$
Unique solution: $f(x) = e^x$

Over the course of this unique, we will learn various techniques to explicitly solve differential equations, and also to learn information from them whether we can solve them or not. First, however, we will consider how to get a differential equation in the first place, by modeling a physical situation.

3 Modeling with differential equations

For the rest of this lecture, we will consider some examples of modeling situations with differential equations. Currently, we are not interested in actually solving these equations.

Example 3.1. Suppose that some rabbits are on a very large island with unlimited food and no predators. Needless to say, they rapidly get to breeding. Let $R(t)$ denote the number of rabbits on the island at time t. How might we model this situation?

One way to think about this is that each rabbit will have some number of kits (baby rabbits) each year, and each kit is quickly added to the breeding population. So the *rate of change* of the number of rabbits on the island should be proportional to the number of rabbits currently on the island. So we can model this as something like the following.

$$R'(x) = 10 \cdot R(x)$$

Here, 10 is a constant which reflects how many kits each rabbit will have in a year. It could easily be 50 or any other number.

Example 3.2. Now suppose that the rabbits are not interested in breeding for some reason, so they don't. On the other hand, the island also includes a single Yeti, which eats 100 rabbits each year. Now how might we model the function $R(t)$?

The number of rabbits decreases as a rate of 100 rabbits per year. So this is modeled by this equation.

$$R'(x) = -100$$

Of course, this equation can't hold forever. Sooner or later the Yeti will eat all of the rabbits.

Example 3.3. The previous example was very unrealistic. Yeti or not, the rabbits are going to keep breeding. How could we model a situation where the rabbits keep breeding full speed, despite the fact that a Yeti is eating 100 of them each year. This situation can be modeled simply by including both the change due to breeding and the change due the the Yeti.

$$R'(x) = 10 \cdot R(x) - 100$$

Example 3.4. Suppose 1000 people are on a small island[1]. One of them has just come up with an extremely funny joke, and he begins telling people. Sooner or later everyone on the island will hear the joke; let $H(t)$ denote the number of people who have heard the joke at time t. How might we model the function $H(t)$?

One approach is to imagine that any time, the people on the island are bumping into each other and telling each other the joke. A new person will learn the joke whenever someone who knows the joke sees someone who does not. The number of such meetings should be proportional to the product of the number of people who know the joke and the number of people who do not. In other words, it should be proportional to $H(t)\,(1000 - H(t))$. So here is a differential equation that could model this situation.

$$H'(t) = C \cdot H(t)\,(1000 - H(t))$$

Here, C is a constant that reflects how frequently the people in the town see each other.

Example 3.5 (Nominal interest). Suppose that you have some money in a bank account, given by $M(t)$ at time t, which receives 2% nominal interest per year, compound continuously. What this means is that the rate at which interest accumulates in the account is $0.02M(t)$ dollars per year. As a differential equation, this could be modeled as follows.

$$M'(t) = 0.02M(t)$$

This is called *nominal interest* since it is not the case that you actually receive 2% of your balance as interest each year – in fact, you receive interest payments throughout the year, at a rate which would add up to 2% at the end of the year if it were constant.

Example 3.6. As a variation on this example, suppose that you have a bank account with 2% nominal interest per year, compounded continuously, but that you also withdraw 1000 per year at a constant rate. This situation could be modeled with the following equation.

$$M'(t) = 0.02M(t) - 1000$$

Without solving this equation, you can try to answer qualitative questions about it. For example, when is it the case that you will be losing money? In fact, you will only be losing money if $M'(t) < 0$, which is the case if and only if $0.02M(t) < 1000$, i.e. $M(t) < 50,000$. So as long as you begin with $50,000$ dollars in your account, you can without 1000 dollar each year without the account balance decreasing. This sort of reasoning will be expanded upon and done in a broader class of situations in the upcoming two lectures.

[1] Don't worry; not all of our examples will take place on islands.