Lecture 28: Separation of variables

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1 Introduction

This lecture begins the study of explicit solutions to differential equations. Unfortunately, most differential equations are rather difficult to solve explicitly, but there are still a number of cases where explicit solutions can be found. Today we consider so-called *separable* differential equations. The autonomous equations that we considered last time are an important example of separable equations. Separable equations are essentially equations that can be written $y' = f(y)g(x)$, i.e. the slope of y can be written as a product of a function of y and a function of x.

The basic technique we consider is to solve separable differential equations by first "separating" them, obtaining two expressions that can be integrated to obtain a new equation that does not involve any derivatives.

An important thread to pay attention to in this discussion is how the various solutions to a differential equation differ from each other. Usually, there will be some constant C that can vary to give the different solutions, but varying C will have rather different effects in different cases.

Although relatively few differential equations are actually separable, many more equations can be solved by first performing some substitution that transforms them into separable equations. This is an important technique that is explored in detail in courses on differential equations; we shall discuss only one or two examples in this course.

The technique of separation will be out first technique to find explicit solutions. In the next class, we shall consider another technique, making use of power series representations of functions.

The reading for today is Gottlieb §31.4. The homework is problem set 27 (which includes weekly problem 25) and a topic outline. You should begin working on weekly problem 28.

2 Equations of the form $y' = f(t)$

Suppose that y is a function of t. I shall sometimes write $y(t)$ and sometimes write simply y for this function; usually $y(t)$ is used when it is important to clearly emphasize what variable y is a function of, whereas y is written to save space and make expressions less crowded. Both notations are common, so you should be flexible.

The easiest type of differential equation to solve explicitly is a differential equation of the form $y'(t) = f(t)$, where f is some function. Indeed, this differential equation is stating precisely that *y is an antiderivative of f*. Such an equation can be solved simply by integration.

$$y(t) = \int f(t)dt$$

Of course, since this is an indefinite integral, it will include a $+C$ term. For example, if $F(t)$ is an antiderivative of $f(t)$, then the conclusion here will be that $y(t) = F(t) + C$. Therefore, there will be many possible solutions; which one is correct is determined by the initial condition.

Example 2.1. Solve the equation $y'(t) = t$, with the initial condition $y(1) = 2$.

By integration, $y = \int t dt = \frac{1}{2}t^2 + C$. Thus $y(1) = \frac{1}{2} + C$, and therefore in order for $y(1)$ to be 2, C must be $\frac{3}{2}$. Therefore the desired solution is $y = \frac{1}{2}t^2 + \frac{3}{2}$.

The technique of separation of variables simply attempts to do the same trick: integrate both sides of the differential equation.

3 The separation technique

This technique of separation of variables is best illustrated by examples.

3.1 Examples

Example 3.1. Consider the equation $y' = y$, where y is a function of t. This can be separated by rewriting it as $\frac{1}{y}y' = 1$, as long as $y \neq 0$ (we will return to this exception in a moment). Now integration both sides with respect to t.

$$\begin{aligned} \frac{1}{y(t)}y'(t) &= 1 \\ \int \frac{1}{y(t)}y'(t)dt &= \int dt \end{aligned}$$

Now, the integral on the left can be evaluated by substitution. Observe that $dy = y'(t)dt$, so the integral is quite easy to transform into an integral with respect to y.

$$\begin{aligned} \int \frac{1}{y}dy &= \int dt \\ \ln|y| &= t + C \end{aligned}$$

This is now an equation that does not involve any derivatives. It is simply necessary to solve it for y.

$$\begin{aligned} |y| &= e^{t+C} \\ y &= \pm e^C e^t \end{aligned}$$

Since e^C can be any positive number $\pm e^C$ can be any number other than zero. The conclusion is that y is some multiple $C_1 e^t$ of the exponential function, where C_1 can be any number other than 0 (here $C_1 = \pm C$).

There is one issue with this discussion: in the very first step, we divided by y, therefore implicitly assuming that $y \neq 0$. Fortunately, this case is easy to deal with separately: since $y = 0$ is an equilibrium solution of this differential equation, if $y(t) = 0$ for any particular t, the function $y(t)$ is equal to 0 for all t.

The final conclusion is that the function y satisfied $y' = y$ if and only if either $y = Ce^t$ for some $C \neq 0$, or $y = 0$. Of course, we may express both cases in one simple form.

$$\begin{aligned} \text{If} \quad & y' = y \\ \text{then} \quad & y = Ce^t \\ \text{where} \quad & C \text{ is any constant.} \end{aligned}$$

Notice that whereas in the previous section, the different solutions to the equation differed by the addition of a constant, here the different solutions vary by multiplication by a constant.

The example above was much more verbose than necessary, in order to show all the steps. The following example is essentially identical, but is done more tersely.

Example 3.2. Consider the equation $y' = 3y$.

This equation is separable. *If* $y \neq 0$, then we can solve it as follows.

$$\begin{aligned} y' &= 3y \\ \int \frac{y'}{y} dt &= \int 3dt \\ \int \frac{1}{y} dy &= \int 3dt \\ \ln|y| &= 3t + C \\ y &= \pm e^{3t+C} \\ &= \pm e^C \cdot e^{3t} \end{aligned}$$

Now, the constancy $\pm e^C$ can be any constant besides 0. Now, at the very beginning we assumed that $y \neq 0$ (in order to divide by y). In case $y(t) = 0$ at any value t, in fact $y(t) = 0$ for all t (this is an equilibrium solution). Summing up, we have the following general solution.

$$y = Ce^{3t}$$

where C can be any constant (positive, negative, or zero).

The two examples considered so far are both autonomous differential equations. This is one of the most important instances where separation is useful, but the technique is considerably stronger, as the following example shows.

Example 3.3. Consider the equation $y' = -\frac{t}{y}$. We can separate this to obtain $y \cdot y' = -t$. Since we are not dividing by anything that could be 0, this is valid without exceptions. Now, we can integrate both sides with respect to t.

$$\begin{aligned} \int y \cdot y' dt &= -\int tdt \\ \int ydy &= -\int tdt \\ \frac{1}{2}y^2 &= -\frac{1}{2}t^2 + C \\ y &= \pm\sqrt{2C - t^2} \end{aligned}$$

So the solution curves to this equation are semicircular arcs.

It is worth noting that the penultimate line in the derivation above is equivalent to the sentence $y^2 + t^2$ *is constant*. In other words, in the graph of any solution curve, the distance to the origin is constant. Therefore this differential equation can be thought of as asserting conservation of energy, where here energy is given by $y^2 + t^2$. This idea – finding an "energy" function of y and t that must be constant on all solution curves – is quite versatile in more sophisticated differential equations (and is important, for obvious reasons, in differential equations arising from physics).

3.2 The technique in general

The technique of separation of variables proceeds in three steps.

1. Reformulate ("separate") the equation so that everything involving y is on one side, and everything involving t is on the other. Keep track of all exceptional cases (where you are dividing by something that could be 0).

2. Integrate both sides of the equation to obtain an equation with no derivatives.

3. Solve the resulting equation to obtain an expression for y.

4. Consider exceptional cases discovered in step 1 to assemble a complete set of solutions.

In symbols, we could describe the technique as follows. A differential equation is called *separable* if it is of the form $y' = f(y)g(t)$, for some functions f, g. It is called separable because the right side can be separated into something that depends on y and something that depends on t.

Step one is to separate the equation, obtaining $\dfrac{y'}{f(y)} = g(t)$. Take note of the fact that this is only valid when $f(y) \neq 0$; this exception will be dealt with later.

Step two is to integrate, obtaining $\displaystyle\int \frac{1}{f(y)}dy = \int g(t)dt$. If $F(y)$ is an antiderivative of $\frac{1}{f(y)}$ and $G(t)$ is an antiderivative of $g(t)$, then the result will be $F(y) = G(t) + C$.

Step three is to solve the resulting equation for y. How this is done will depend on the function F, but in general it will be something like $y = F^{-1}(G(t) + C)$. This gives most of the solutions to the differential equation.

Step four is to account for the exceptions from step one, namely the cases where $f(y) = 0$. In fact, any value of y such that $f(y) = 0$ corresponds to a constant solution (i.e. an *equilibrium solution*). So add to the list of solutions from the previous step all constant solutions corresponding to values of y where $f(y) = 0$.

3.3 Differential notation

There is another notation that is frequently used when solving differential equations by separation. It is illustrated below: consider the differential equation $y' = ty$, and use the notation $\dfrac{dy}{dt}$ for y'. The idea is the treat $\frac{dy}{dt}$ as a fraction, and separate it into two pieces.

$$
\begin{aligned}
\frac{dy}{dt} &= ty \\
\frac{1}{y}dy &= tdt \\
\int \frac{1}{y}dy &= \int tdt \\
\ln|y| &= \frac{1}{2}t^2 + C
\end{aligned}
$$

The point here is that, although the expression $\dfrac{dy}{dt}$ is not actually a fraction, we can frequently manipulate it as if it were a fraction and obtain correct results. Separation of variables is one such case. This notation, though slightly imprecise, is fairly intuitive and often convenient.

4 Examples

Here are a couple more examples of solving differential equations by separation of variables.

Example 4.1. Consider the logistic differential equation $y' = y(1-y)$, often used as a simple model of population dynamics. So long as y is neither 0 nor 1, this can be separated and solved by integration. The necessary integration technique is the technique of partial fractions.

$$\begin{aligned}
\frac{1}{y(1-y)}y' &= 1 \\
\int \frac{1}{y(1-y)}dy &= \int dt \\
\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy &= \int dt \\
\ln|y| - \ln|1-y| &= t + C \\
\ln\left|\frac{y}{1-y}\right| &= t + C \\
\frac{y}{1-y} &= \pm e^{t+C} \\
&= \pm e^C e^t
\end{aligned}$$

Now by redefining the arbitrary constant, this last expression could be written Ce^t, where C is now any constant besides 0. Now it is necessary to solve for y.

$$\begin{aligned}
\frac{y}{1-y} &= Ce^t \\
y &= Ce^t(1-y) \\
y(1+Ce^t) &= Ce^t \\
y &= \frac{Ce^t}{1+Ce^t}
\end{aligned}$$

Thus, this gives a family of solutions to the original differential equation: $y = \dfrac{Ce^t}{1+Ce^t}$, where C is a constant other than 0. Now, this analysis has left out two equilibrium solutions: $y = 0$ and $y = 1$. The constant solution $y = 0$ can be included simply by allowing C to be 0, but the solution $y = 1$ is still an exceptional case.

So the complete list of solutions to $y' = y(1-y)$ is: either $y = 1$ or $y = \frac{Ce^t}{1+Ce^t}$ for some constant C.

Observation 4.2. The solution $y = \frac{Ce^t}{1+Ce^t}$ could also be written as $\frac{1}{1+C^{-1}e^{-t}}$ (for $C \neq 0$). Alternatively, replacing C with C^{-1}, this gives another description of a family of solutions: $y = \frac{1}{1+Ce^{-t}}$. Now, taking $C = 0$ gives one equilibrium solution $y = 1$, while the other equilibrium solution $y = 0$ comes, intuitively, from taking $C = \infty$, although this is certainly not a mathematically justified thing to say.

Example 4.3. Find the solution of $y' = y(1-y)$ such that $y(0) = 2$.

From the previous example, all non-equilibrium solutions of this differential equation have the form $y = \frac{Ce^t}{1+Ce^t}$, for some C. We simple need to determine what C should be, given this initial condition. Setting $t = 0$, we obtain the equation $y(0) = 2 = \frac{C}{1+C}$. Solving this:

$$
\begin{aligned}
2 &= \frac{C}{1+C} \\
2+2C &= C \\
C &= -2.
\end{aligned}
$$

Therefore the desired solution function is $y = \frac{-2e^t}{1-2e^t}$. Alternatively, this could be written $y = \frac{2}{2-e^{-t}}$, by dividing the numerator and denominator by $-e^t$.

Notice that, as we should expect from qualitative analysis, this solution approaches the stable equilibrium $y = 1$ as t approaches infinity.

Example 4.4. Consider the equation $y' = ty$. What is the solution such that $y(0) = 2$?

As shown in the previous subsection, either $y = 0$ or $\ln|y| = \frac{1}{2}t^2 + C$. In the latter case, this can be solved as $y = \pm e^C e^{t^2/2}$. Since $\pm e^C$ can be any constant besides 0, and since $y = 0$ is the equilibrium solution that is missed by dividing by y, the family of all solutions is described by $y = Ce^{t^2/2}$, where C is any constant.

Imposing the initial condition $y(0) = 2$ means that $C = 2$, so the solution satisfying this initial condition is $y = 2e^{t^2/2}$.

Example 4.5. Consider the equation $\frac{dy}{dx} = x(2-y)$, which was mentioned in problem 4 of problem set 25. On that assignment, the solution was found from a multiple-choice list, but we now have the necessary techniques to solve it directly.

First, separate the equation. Observe that since this involves dividing by $2-y$, we must remember to also include the equilibrium solution $y = 2$ in the final list of solutions. Separating and integrating gives:

$$
\begin{aligned}
\int \frac{1}{2-y} dy &= \int x dx \\
-\ln|2-y| &= \frac{1}{2}x^2 + C \\
\ln|2-y| &= -\frac{1}{2}x^2 - C \\
2-y &= \pm e^{-\frac{1}{2}x^2 - C} \\
y &= 2 \pm e^{-C} e^{-\frac{1}{2}x^2}
\end{aligned}
$$

Now, $\pm e^{-C}$ can be any constant besides 0, so we may rewrite this family of solutions as $y = 2 + Ce^{-x^2/2}$, where C is any constant besides 0. The only missing solution is the equilibrium $y = 2$, which can be included in this family by simply allowing C to be 0.

Therefore the complete set of solutions is $y = 2 + Ce^{x^2/2}$, where C is any constant. Indeed, this is the same family of solutions as was found by guesswork in the homework problem.

5 Transforming to separable equations by substitution

This section is a brief illustration of a technique that is often useful for solving differential equations that are not separable. We shall not discuss this technique in any detail or generality, but it is good to see an example of it to know that such techniques will sometimes help.

Consider the following differential equation. Here, y is a function of x.

$$
y' = \frac{y^2 - xy}{x^2}
$$

In its current form, this equation is not separable. The remainder of this section will consist of solving this differential equation by making a substitution that produces a separable equation.

Notice that the equation can be rewritten.

$$y' = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

This suggests that it may be simpler to study the function $\frac{y}{x}$, rather than the function y itself. Therefore, do a substitution: let $u = \frac{y}{x}$. Then $y' = u^2 - u$. But we need to write a differential equation for u, not y. Towards this end, note that $y = ux$, therefor $y' = u'x + u$. Therefore:

$$\begin{aligned} y' &= u^2 - u \\ u'x + u &= u^2 - u \\ u' &= \frac{u^2 - 2u}{x} \end{aligned}$$

But now this is a separable differential equation for the function u, which can be solved in the usual manner. Away from the equilibrium solutions $u = 0$ and $u = 2$,

$$\begin{aligned} \frac{1}{u^2 - 2u} u' &= \frac{1}{x} \\ \int \frac{1}{u^2 - 2u} du &= \int \frac{1}{x} dx \\ \int \left(\frac{1/2}{u - 2} - \frac{1/2}{u} \right) du &= \int \frac{1}{x} dx \\ \frac{1}{2} \ln|u - 2| - \frac{1}{2} \ln|u| &= \ln|x| + C \\ \ln\left|\frac{u - 2}{u}\right| &= 2\ln|x| + 2C \\ \frac{u - 2}{u} &= \pm e^{2\ln|x| + 2C} \\ 1 - \frac{2}{u} &= \pm e^{2C}|x|^2 \\ \frac{2}{u} &= 1 \pm e^{2C}x^2 \\ u &= \frac{2}{1 \pm e^{2C}x^2} \end{aligned}$$

Now, $\pm e^{2C}$ can be any constant besides 0, so we may take it to be our arbitrary constant. In fact, the value 0 is acceptable as well, since it leads to the equilibrium solution $u = 2$.

Thus, the total list of solutions to the differential equation $u' = \frac{u^2 - 2u}{x}$ is $u = \frac{2}{1 + Cx^2}$ (where C is any constant) and $u = 0$.

Translating these solutions back into solutions of the original equation via $y = ux$, the list of all solutions to the equation $y' = \frac{y^2 - xy}{x^2}$ is $y = \dfrac{2x}{1 + Cx^2}$ and $y = 0$.