Lecture 30: Second order homogeneous equations I

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1 Introduction

This lecture introduces a very particular class of second order differential equations. The main physical interpretation for these equations comes from oscillating systems, such as springs or pendulums. The main distinguishing feature of these equations is that they are *linear*, in a sense which will be described.

We mainly study these equations due to the behavior that they illustrate. On the one hand, they are a simple but nontrivial example of second-order equations. On the other, much of the behavior of systems of multiple differential equations can be seen in the behavior of these equations, as we shall see in the last couple lectures of this course.

Today, we shall consider three simple examples, and construct all solutions to these three examples. The techniques used will be rather different in each case. In the following class, we shall bring these techniques together to give a general method.

The reading for today is the handout "Second-order homogeneous differential equations" (under "reading for the course"). The homework is problem set 29 (which includes weekly problem 28) and a topic outline.

2 Statement of the problem

This lecture and the one that follows will focus entirely on differential equations of the following form. Here x is a function of a variable t, and b, c are constants.

$$x''(t) + bx'(t) + cx(t) = 0 \tag{1}$$

For the rest of today, I will often suppress the (t) for clarity; but x will always be assumed to be a function of t.

Such equations are referred to as *second-order homogeneous linear differential equations with constant coefficients,* but you don't need to worry about this mouthful of words. Briefly, though, I will explain the adjectives: *second order* means there are second derivatives involved; *homogeneous* means that there is a zero on the right side; *linear* means that the left side is a sum of derivatives of the function multiplied by some coefficients; *with constant coefficients* means that these coefficients are constant.

For brevity, I shall refer to all equations of the form of equation 1 (i.e., as the constants b, c vary) as "second-order homogeneous equations," as in the title.

Our goal is to fully describe all solutions to any such equation. We shall complete this task in the following lecture. For today, I will focus on the following three examples, which will be solved completely. These examples are chosen because they illustrate the three main type of behavior that these equations can have.

I. $x'' = 0$

II. $x'' + x = 0$

III. $x'' + 3x' + 2x = 0$

First I will state the general sort of form that the solutions to these equations will have, and then the solutions will be presented one by one.

3 Initial value problems and general solutions

Recall that an *initial value problem* for a second-order differential equation contains three data: a differential equation, an initial position (e.g. $x(0)$) and an initial velocity (e.g. $x'(0)$). The basic fact is that every initial value problem has a *unique* solution.

Given this fact, we have a way to tell when we have found all possible solutions to a differential equation: if we can find a family of solutions that can solve every possible initial value problem, then this family includes all possible solutions of the differential equation. Such a family is called a *general solution*[1] for a differential equation. We shall see three examples today, by finding the general solutions for equations I, II, and III.

There is another very helpful feature of all second-order homogeneous equations (common to all so-called *linear* differential equations), which makes it much easier to find the general solution.

Fact. Suppose that $x(t) = f(t)$ and $x(t) = g(t)$ are two nonzero solutions of $x'' + bx' + cx = 0$ (for some particular values of b and c). These solutions are called *independent* if neither is a constant multiple of the other. If these two solutions are independent, then the general solution of the differential equation is $C_1 f(t) + C_2 g(t)$, where C_1, C_2 are arbitrary constants.

There are two things at work in this fact: first, the sum of two solutions is another solutions, and second, any constant multiple of a solution is another solution. It is these two things which explain why these differential equations are called *linear*. Because of them, it is always true that $C_1 f(t) + C_2 g(t)$ is a solution, for any C_1, C_2. The fact that f and g are independent simply guarantee that, by varying C_1 and C_2, and initial value problem can be solved.

Using this fact, we can now described the general solutions to equations I, II, and III.

4 General solution of I

Equation I is intended as a warm-up. Indeed, it can be solved without any special techniques, simply by integration.

$$\begin{aligned}
x''(t) &= 0 \\
x'(t) &= \int 0 dt \\
&= C_1 \\
x(t) &= \int x'(t) dt \\
&= \int C_1 dt \\
&= C_1 t + C_2
\end{aligned}$$

[1] N.B. a "general solution," despite the name, is not a single solution, but rather a *family* of solutions.

In terms of the fact from the previous section: $x(t) = t$ and $x(t) = 1$ are two *independent* solutions of equation I. Therefore, the general solution simply consists of linear combinations of these two solutions: $C_1 t + C_2$. Indeed, this can solve any initial value problem, as the following example illustrates.

Example 4.1. Solve the equation $x'' = 0$ subject to the initial conditions $x(0) = 3, x'(0) = -1$.
To do this, simple note that if $x(t) = C_1 t + C_2$, then $x(0) = C_2$ and $x'(0) = C_1$. So we must take $C_1 = -1, C_2 = 3$. The desired solution is $x(t) = -t + 3$.

5 Ideal springs and the general solution of II

Now consider equation II.

$$x''(t) + x(t) = 0$$

Alternatively, this could be written $x'' = -x$. In words: *the acceleration is proportional to the position, but in the opposite direction.* In fact, this equation is a good model for the motion of a weight attached to a spring (at least on short time scales, before friction has taken its toll). It is also a good model for the motion of a pendulum[2]. The reason that this models this type of motion is that the further a spring is stretched, the harder is pulls back in the opposite direction (and similarly when it is compressed, it pushes back proportionally to how far it is compressed). This relationship is called *Hooke's law.* In Newtonian mechanics, the force applied to an object is proportional to its acceleration, thus we obtain the equation $x'' = -kx$, where k is some positive constant.

How could we solve equation II? One option is to try to guess the solution. If you imagine how a weight on a spring will move, you will conclude that it will oscillate back and forth, so a good guess is $x(t) = \sin x$ or $x(t) = \cos x$. In fact, both of these are solutions, and the are independent. Therefore the general solution is easy to write down, simply from this guesswork.

$$x(t) = C_1 \sin x + C_2 \cos x$$

Here, C_1, C_2 are two arbitrary constants. Observe that for this solution of the differential equation, $x(0) = C_2$ and $x'(0) = C_1$. So clearly any initial value problem can be solved by some solution of this form; this is indeed the general solution.

Observation 5.1. This discussion has an interesting consequence: any solution of equation II should look like an oscillating function, i.e. a sine wave. Thus linear combinations of $\sin x$ and $\cos x$, such as $x(t) = \sin x + \cos x$, must in fact be sinusoidal functions. This can also be seen, with somewhat more difficult, using trigonometric identities. For example, $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.

In fact, this same type of guesswork will furnish solutions to equations like II, with different constants.

Example 5.2. Find the solution to $x''(t) + 4x(t) = 0$ such that $x(0) = -1$ and $x'(0) = 1$.
We expect the solution to be a sine wave of some kind. However, $\sin x$ is not a solution, However, we can get a solution by introducing a scale factor. Notice that if $x(t) = \sin(2x)$, then $x'(t) = 2\cos(2x)$ and $x''(t) = -4\sin(2x)$, which is $-4x(t)$. Hence $\sin(2x)$ is a solution to this differential equation. By similar reasoning, so is $\cos(2x)$. So the general equation is $x(t) = C_1 \sin(2x) + C_2 \cos(2x)$. For such a solution, $x(0) = C_2$ and $x'(0) = 2C_1$. Hence we can take $C_1 = \frac{1}{2}$ and $C_2 = -1$. The desired solution is therefore $x(t) = \frac{1}{2}\sin(2x) - \cos(2x)$.

From this example, you should be able to imagine how to solve other very similar equations, such as $x'' + 16x = 0$.

[2] Actually, a slightly better model for a pendulum is $x'' = -\sin x$, but since $\sin x \approx x$ for small value of x (by linear approximation), the given equation works well for a pendulum that is not displaced too far.

6 Damped springs and the general solution of III

Now consider equation III.

$$x''(t) + 3x'(t) + 2x(t) = 0$$

Alternatively, this could be written as $x''(t) = -2x(t) - 3x'(t)$. Thinking physically, this means that an object has a force acting on it that is proportional to its position (in the opposite direction), and another force that is proportional to the velocity (in the opposite direction from the velocity). In fact, this is a fairly good model for what is called a *damped spring*, i.e. a spring that is being impeded by various sorts of friction (perhaps because the mass is moving through a viscous fluid, for example). The term $3x'(t)$ in this equation is called the *damping term* for this reason.

How might we expect the solutions to such an equation to look? One guess is that they will still oscillate, but that the waves will get smaller and smaller in magnitude as time goes on and energy is dissipated. This is what you would see if you run a pendulum clock for a long time, for example. We shall see next time that this is the behavior that is seen in many cases.

As it happens, though, other behavior is possible as well. In this case, the damping term is rather large. A spring that is this heavily damped could be imagined to be immersed in molasses, for example. In this situation, you can imagine that perhaps the mass will simply be slowly pulled back to equilibrium, slowing as it reaches it. Such a situation is called an *over-damped spring*.

It turns out that equation III models an over-damped spring, and we can solve it with a little guesswork. Perhaps there is a solution to this equation that looks like exponential decay. So we could guess that $x(t) = e^{-rt}$ is a solution for some value r. To decide which value of r to use, just plug this function into the equation.

$$\begin{aligned} x(t) &= e^{-rt} \\ x'(t) &= -re^{-rt} \\ x''(t) &= r^2 e^{-rt} \\ x''(t) + 3x'(t) + 2x(t) &= (r^2 - 3r + 2)e^{-rt} \end{aligned}$$

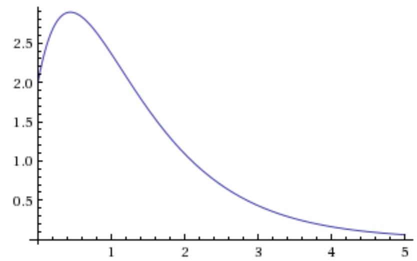
Therefore, we should choose a value of r such that $r^2 - 3r + 2 = 0$. Fortunately, we know how to find such r: factor the equation to obtain $r^2 - 3r + 2 = (r-1)(r-2)$; the possible values of r are 1 an 2. So in fact we obtain not one but two possible solutions to equation III this way: e^{-t} and e^{-2t}. So the general solution is the following.

$$x(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Notice that all such solutions decay to 0 as t goes to infinity. How quickly they decay will depend on the initial conditions.

Example 6.1. Solve $x'' + 3x' + 2x = 0$ subject to the initial conditions $x(0) = 2, x'(0) = 5$.

For $x(t) = C_1 e^{-t} + C_2 e^{-2t}$, we must have $x(0) = C_1 + C_2$ and $x'(0) = -C_1 - 2C_2$. Thus to meet the given initial conditions, we must have $C_1 + C_2 = 2$ and $C_1 + 2C_2 = -5$. Solving this pair of equations gives $C_1 = 9, C_2 = -7$. Therefore the desired solution is $x(t) = 9e^{-t} - 7e^{-2t}$. A plot of this function is shown below.



Indeed, this graph matches the specified behavior: it begins with a sharp upward motion, but it then pulled back to 0, slowing as it reaches it due to the damping.

7 The characteristic equation and exponential solutions

The guesswork that solved equation III is useful in a board range of situations (and it will, in fact, form the core of our work in the next class). The basic fact is that for any equation of the form of equation 1, we can look for exponential solutions, and we will find them when we can solve a certain quadratic equation.

$$\begin{array}{rl} \text{If} & x(t) = e^{rt} \\ \text{Then} & x''(t) + bx'(t) + cx(t) = (r^2 + br + c)e^{-rt}. \end{array}$$

It follows from this that $x = e^{rt}$ is a solution to $x'' + bx' + cx = 0$ if and only if r is a solution to $r^2 + br + c = 0$. For this reason, the equation $r^2 + br + c = 0$ is called the *characteristic equation* of the differential equation $x'' + bx' + cx = 0$. If it can be solved to obtain real roots, then we will obtain solutions to the differential equation.

Example 7.1. Find the general solution to the equation $x''(t) - 4x(t) = 0$.

The characteristic equation of this differential equation is $r^2 - 4 = 0$. This equation can two solutions: $r = 2$ and $r = -2$. Therefore, $x(t) = e^{2t}$ and $x(t) = e^{-2t}$ are two solution functions (you can easily verify that they solve the differential equation).

Since we have two independent solutions, we have the general solution: $x(t) = C_1e^{2t} + C_2e^{-2t}$.

Why doesn't this method work to solve all equations of the form of equation 1? The reason is that the characteristic equation doesn't always have two different real roots. I leave it to you to see what happens if you try to apply this method to equations I and II. The result is intriguing, and will be the starting point of the discussion next time.