Limits and infinity

Nathan Pflueger

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This brief note explains how limits involving infinity can meaningfully be defined, using one notion that nicely unifies limits involving infinity and limits not involving infinity. The basic idea is that of a *neighborhood*, which is the basic notion in the mathematical field called topology. This is meant purely to satisfy any curiosity; it is not required material for the course. It is possible that it will clear up some of your thinking about limits, however.

Reminder. For math 1B, if a given limit is equal to $\pm\infty$ (in the sense described here), we shall still say that it diverges, or perhaps "diverges to infinity." This language is not universal, but it is what we shall use.

All number used in this note will come from the *extended real line*, denoted $[-\infty, \infty]$, which consists of all the usual real numbers, plus two infinite quantities, $-\infty$ and $+\infty$. Thus if I say that "x is an element of $[-\infty, \infty]$," I mean that x is either a real number, $+\infty$, or $-\infty$.

The usual definition of limit can be stated informally in the following way: the statement "$\lim_{x \to b} f(x) = L$" means that for any margin of error around L, $f(x)$ remains within that margin of error as long as x is sufficiently close to b. Here, "margin of error" and "sufficiently close to" both describe the same sort of idea: a margin of error specifies a little neighborhood of L where we want to claim that $f(x)$ must fall. In order for $f(x)$ to fall there, x must be "sufficiently close to" b, which might be stated by saying that there is some neighborhood of b so that as long as x is in this neighborhood, $f(x)$ will be in the desired margin of error. Making this precise, we arrive at the following definition.

Definition 1. For any elements b and L of the extended real line $[-\infty, \infty]$, the notation $\lim\limits_{x \to b} f(x) = L$ denotes the following statement.

For every	neighborhood $\mathcal{U}$ of L,
there exists	a neighborhood $\mathcal{V}$ of b
such that	whenever $x \neq b$ and x is in $\mathcal{V}$, $f(x)$ is in $\mathcal{U}$.

If such an L exists, we say that $\lim_{x \to b} f(x)$ exists. Otherwise, we say that this limit does not exist.

This definition may look somewhat abstract and hard to parse, but it is actually a very simple idea. It just expressed the informal notion: $f(x)$ will be sufficiently close to L as long as x is sufficiently close to b. The terminology of neighborhoods merely makes this precise.

Of course, definition 1 is incomplete until I tell you what a neighborhood is. The beautiful thing is that, although two different definition of neighborhood are needed for finite and infinite values, only one definition of limit is needed. Here are the relevant definitions.

Definition 2. For any number x in $(-\infty, \infty)$, a *neighborhood* of x is an open interval $(x - \epsilon, x + \epsilon)$, where ϵ is a (usually small) positive number. In other words, a neighborhood is the set of points within some margin of error of x.

Definition 3. For $x = +\infty$, a *neighborhood* of x is an interval (C, ∞), where C is a real number. For $x = -\infty$, a *neighborhood* of x is an interval $(-\infty, C)$, where C is a real number.

What these definitions achieve can be stated as follows: restricting a number to a neighborhood of x amounts to either restricting it to be within some margin of error of x (if x is finite), restricting it to be sufficiently large (if $x = \infty$), or restricting it to be sufficiently small (if $x = -\infty$).

You have likely seen the so-called "epsilon-delta" definition of limit. Definition 1 is equivalent to this definition; it may be interesting to work out precisely why.

In order to illustrate why, for these definitions of neighborhood, definition 1 gives a good notion of limit, I shall state, informally, what definition 1 amounts to in in some specific cases.

Statement	Informal meaning
$\lim_{x\to 1} f(x) = 0$	For x sufficiently close to 1, $f(x)$ is within any chosen margin of error of 0.
$\lim_{x\to 1} f(x) = \infty$	For x sufficiently close to 1, $f(x)$ exceeds any chosen lower bound.
$\lim_{x\to \infty} f(x) = 1$	For x sufficiently large, $f(x)$ is within any chosen margin of error of 1.
$\lim_{x\to +\infty} f(x) = +\infty$	For x sufficiently large, $f(x)$ exceeds any chosen lower bound.

Observation 1. To define limits "from the left" or "from the right", it is merely necessary to define "left neighborhoods" and "right neighborhood." For example, a left-neighborhood of x is an interval like $(x-\epsilon, x)$, for ϵ a (usually small) positive number.

Observation 2. The following pairs of statements are equivalent. This gives another way to define limits involving infinity.

$$\lim_{x\to\infty} f(x) \quad \Leftrightarrow \quad \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$$

$$\lim_{x\to-\infty} f(x) \quad \Leftrightarrow \quad \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = L$$

$$\lim_{x\to b} |f(x)| = +\infty \quad \Leftrightarrow \quad \lim_{x\to b} \frac{1}{f(x)} = 0$$

The absolute value signs are needed on the left side of the third equivalence for technical reasons; the limit of $\frac{1}{f(x)}$ being 0 doesn't indicate anything about the sign of $f(x)$; the limit of $f(x)$ could be $+\infty$, $-\infty$, or neither (in case the sign oscillated between positive and negative).