Summation Notation

Nathan Pflueger

12 October 2011

This note gives some examples of conversion between Σ and $\cdots$ notation for Taylor series.

I elaborate here in *much* more detail than you should ever write down; my aim here is to describe how you might go about finding the correct expression on your scratch paper. Of course, eventually all this will become so habitual that you will need to write nearly nothing down.

1 The exponential function

The Maclaurin series of the exponential function e^x (i.e. the Taylor series with center 0) in the most basic Taylor series, in the sense that all others basically resemble it. I will describe the notation here in some detail to illustrate the general picture.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$$

Here the ellipsis $\cdots$ indicates that the sequence of terms is meant to continue "as before." In order to make sense of this, it is necessary to recognize what pattern is meant to proceed as before. In this case, the pattern is that each term is the derivative of term that follows it, i.e. each term is the integral (from 0 to x) of the term that precedes it. So this coefficients can be written more suggestively by not fully multiplying out the denominators.

$$e^x = 1 + \frac{1}{1}x + \frac{1}{1\cdot 2}x^2 + \frac{1}{1\cdot 2\cdot 3}x^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \cdots$$

Now the pattern is clear: the coefficient of each term is the reciprocal of the product of some number of consecutive numbers. Fortunately, there is a standard notation for this, which is the factorial symbol.

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

Now, to express this in Σ notation, it is first necessary to describe the terms of the sum in a "general" way. In this case, the terms all look like this: $\frac{1}{\bullet!}x^{\bullet}$. Here, both bullets are standing for a number; in fact, they stand for the same number. Let's give it a name: k. We could just as well have used n or i. So the general term looks like this: $\frac{1}{k!}x^k$. Note something nice that happens in this case: if we plug in $k = 0$ and use the standard conventions $x^0 = 1, 0! = 1$, then all terms (including the first one) look the same.

$$e^x = \frac{1}{0!}x^0 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

Note that sometimes, one makes the general term of the sum totally explicit by including it after the ellipsis, as below. Again, to emphasize: any letter can be used; it doesn't have to be k.

$$
\begin{aligned}
e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{k!}x^k + \cdots \\
e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots \\
e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{i!}x^i + \cdots
\end{aligned}
$$

The last ellipsis is important; it indicates that the sum continues on to infinity.

Now, since the general term looks like $\frac{1}{k!}x^k$, we know that the series will be written in the form $\sum \frac{1}{k!}x^k$, although we will need to decorate the $\sum$ to make it fully specified. In particular, it is necessary to specify which values of k will be used. In this context, the first term is $\frac{1}{0!}x^0$, which is $\frac{1}{k!}x^k$ for $k = 0$. Therefore the first term is $k = 0$. So the notation can be updated to $\sum_{k=0} \frac{1}{k!}x^k$. Finally, it is necessary to supply the superscript, which tells the largest value of k that will be used. In this case, there is not largest value, which is indicated with the superscript ∞. Thus the final form is the following.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

Again, of course, the choice of the letter k was totally irrelevant. The following two notations are exactly equivalent.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!}x^i$$

Finally, note that we didn't necessarily need to recognize the initial term as $\frac{1}{0!}x^0$. We could just as well have left it alone, and only collapsed the other terms into the Σ. In this case, the smallest value of k that is used is $k = 1$. In this case, recall what the notation would signify.

$$\sum_{k=1}^{\infty} \frac{1}{k!}x^k = \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

Therefore, this notation can account for all terms in the Taylor series, starting from the second one. Thus to denote the whole series, we can write the following.

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}x^k$$

2 Common steps

The discussion of the series for the exponential function displayed the main steps involved in converting a Taylor series to Σ notation. Here is a summary of the steps employed, together with brief notes of how they were manifested in this case.

1. Identify what changes in each successive term (denominators multiplied by some number; exponents increased by one).

2. Identify the places where these changes take place in the expression (each term looks like $\frac{1}{\bullet}x^{\bullet}$).

3. Choose an index variable to identify each term with (k, or n, or i).

4. Identify each "changing place" in the expression, and find a suitable way to express it in terms of this variable. The result is the "general term" expressed in terms of a variable ($\frac{1}{k!}x^k$).

5. Identify the subscript of the Σ, which specifies what the index variable is called, and what its *smallest value* is ($k = 0$, or $k = 1$ if the constant term is left outside).

6. Identify the largest value that the index variable must take. If there are infinitely many terms, take this to to be ∞. This becomes the superscript of the Σ.

3 A Taylor series for $\frac{1}{x}$

Let us carry out similar steps in an example: the Taylor series with center $c = 2$ of the function $f(x) = \frac{1}{x}$.

The first step is to compute the derivatives of this function, and evaluate them at the center. Note that I am not fully simplifying all these terms; in general, this will tend to make it easier to identify the pattern.

$$\begin{array}{rr} f(x) = \frac{1}{x} & f(2) = \frac{1}{2} \\ f'(x) = -\frac{1}{x^2} & f'(2) = -\frac{1}{2^2} \\ f''(x) = 2\frac{1}{x^3} & f''(2) = 2\cdot\frac{1}{2^3} \\ f'''(x) = -2\cdot 3\frac{1}{x^4} & f'''(2) = -2\cdot 3\cdot\frac{1}{2^4} \\ f^{(4)}(x) = 2\cdot 3\cdot 4\frac{1}{x^5} & f^{(4)}(2) = 2\cdot 3\cdot 4\cdot\frac{1}{2^5} \end{array}$$

Therefore the Taylor series will begin as follows.

$$\frac{1}{2} - \frac{1}{2^2}(x-2) + 2\cdot\frac{1}{2^3}\frac{1}{2!}(x-2)^2 - 2\cdot 3\cdot\frac{1}{2^4}\frac{1}{3!}(x-2)^3 + 2\cdot 3\cdot 4\cdot\frac{1}{2^5}\frac{1}{4!}(x-2)^4 - \cdots$$

First of all, notice that the factorials in the denominator nicely cancel the constant factors in front.

$$\frac{1}{2} - \frac{1}{2^2}(x-2) + \cdot\frac{1}{2^3}(x-2)^2 - \frac{1}{2^4}(x-2)^3 + \frac{1}{2^5}(x-2)^4 - \cdots$$

Now, notice the following structure emerging.

- At each step, the sign changes (the series is *alternating*).
- At each step, the power of two in the denominator increases by 1.
- At each step, the exponent of $(x-2)$ increases by 1.

It is starting to appear that the general term will always look something like this:

$$\pm\frac{1}{2^{\bullet}}(x-2)^{\bullet}$$

Here, each bullet signifies some number, changing with each term.

Now let's choose a variable. Let's use n this time, and assume this it is the exponent in the general term.

$$\pm\frac{1}{2^{\bullet}}(x-2)^n$$

Now, we need to express the other mystery symbols $\bullet, \pm$ in terms of n. This is not too tricky: just look at a single example, such as this term.

$$+\frac{1}{2^5}(x-2)^4$$

Here, the number before the factorial is one more than the exponent. Since the exponent is n, this should be $n+1$. So the general term should look like this.

$$\pm\frac{1}{2^{n+1}}(x-2)^n$$

The only remaining thing to iron out is the alternating sign. Usually one of two things is done here: either include a term $(-1)^n$ or $(-1)^{n+1}$ (or $(-1)^{n-1}$). Which you use can be discovered by just looking at one term: when $n=4$, the sign is $+$. So we should use $(-1)^n$. So here is the general term.

$$(-1)^n\frac{1}{2^{n+1}}(x-2)^n$$

Or, more compactly:

$$\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

Thus, the notation for the series will look like this.

$$\sum\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

We just need to decorate the $\sum$. Doing this consists of finding the smallest value of n we wish to include, and the largest. There is no largest, so the superscript will be ∞. As for the smallest, the smallest term is

$$\frac{1}{2}=(-1)^0\frac{1}{2^1}(x-2)^0$$

This term corresponds to the case $n=0$. Therefore, adding these decorations, here is the series in $\sum$ notation.

$$\sum_{k=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

4 A Taylor series for $\cos x$

Suppose that we compute the Taylor series for $\cos x$, with center $c=\pi$. Then (omitting some calculations) the first several terms look like this.

$$-1+0\cdot(x-\pi)+1\cdot\frac{1}{2!}(x-\pi)^2+0\cdot\frac{1}{3!}(x-\pi)^3-1\cdot\frac{1}{4!}(x-\pi)^4+0\cdot\frac{1}{5!}(x-\pi)^5+\cdots$$

In this case, it is convenient to throw out the terms that are 0. This leaves behind:

$$-1+\frac{1}{2!}(x-\pi)^2-\frac{1}{4!}(x-\pi)^4+\cdots$$

Here are some pertinent features.

- The series alternates signs.
- The exponent increases by two each time.
- The argument of the factorial increases by two each time.

We guess that the general term should look something like this: $\pm\frac{1}{\bullet!}(x-\pi)^{\bullet}$. Choose a variable: say k this time. Then we don't want to take the exponent of $(x-\pi)^k$, since only even powers are present. Instead, note that we can express every even number as $2k$. So write the general term like $\pm\frac{1}{\bullet!}(x-\pi)^{2k}$. In fact, the argument to the factorial is always the same as the exponent, so this can be written $\pm\frac{1}{(2k)!}(x-\pi)^{2k}t$. As for the alternating sing, just look at one example: $+\frac{2!}{(}x-\pi)^2$. Here $k=1$ (since $2k=2$), and the sign is positive. So $(-1)^k$ won't work; let's use $(-1)^{k+1}$ instead. So the general term is $(-1)^{k+1}\frac{1}{(2k)!}(x-\pi)^{2k}$. The first term, -1, comes from taking $k=0$. So we get the subscript $k=0$. The superscript is ∞, since there is no final term. So the Taylor series we desire is the following.

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k)!}(x-\pi)^{2k}$$